UDC 539.12

POLYADIC SYSTEMS, REPRESENTATIONS AND QUANTUM GROUPS

S.A. Duplij*
Center for Mathematics, Science and Education Rutgers University,
118 Frelinghuysen Rd., Piscataway, NJ 08854-8019
E-mail: duplij@math.rutgers.edu, http://homepages.spa.umn.edu/~duplij
Received 20 May 2012

Polyadic systems and their representations are reviewed and a classification of general polyadic systems is presented. A new multiplace generalization of associativity preserving homomorphisms, a 'heteromorphism' which connects polyadic systems having unequal arities, is introduced via an explicit formula, together with related definitions for multiplace representations and multiactions. Concrete examples of matrix representations for some ternary groups are then reviewed. Ternary algebras and Hopf algebras are defined, and their properties are studied. At the end some ternary generalizations of quantum groups and the Yang-Baxter equation are presented.
KEY WORDS: n-ary group, Post theorem, commutativity, homomorphism, group action, Yang-Baxter equation

ПОЛИАДИЧЕСКИЕ СИСТЕМЫ, ПРЕДСТАВЛЕНИЯ И КВАНТОВЫЕ ГРУППЫ

С.А. Дуплий
Центр математики, науки и образования, университет Ратгерса, Пискатавэй, 08854-8019, США

Приведен обзор полиадических систем и их представлений, дана классификация общих полиадических систем. Построены многоместные обобщения гомоморфизмов, сохраняющие ассоциативность. Определены мультидействия и мультиместные представления. Приведены конкретные примеры матричных представлений для некоторых тернарных групп. Определены тернарные алгебры и Хопф алгебры, изучены их свойства. В заключение, предствлены некоторые тернарные обобщения квантовых групп и уравнения Янга-Бакстера.
КЛЮЧЕВЫЕ СЛОВА: n-арная группа, теорема Поста, коммутативность, гомоморфизм, групповое действие, уравнение Янга-Бакстера

ПОЛІАДИЧНІ СИСТЕМИ, ПРЕДСТАВЛЕННЯ І КВАНТОВІ ГРУПИ

С.А. Дуплій
Центр математики, науки та освіти, університет Ратгерсу, Піскатавєй, 08854-8019, США

Зроблено огляд поліадичних систем та їх представлень, дана класифікація загальних поліадичних систем. Побудовані багатомісні узагальнення гомоморфізмів, що зберігають асоціативність. Визначені мультидії і мультимісні представлення. Наведені конкретні приклади матричних представлень для деяких тернарних груп. Визначені тернарна алгебра і алгебри Хопфа, вивчені їх властивості. На закінчення, предствлені деякі тернарні узагальнення квантових груп та рівняння Янга-Бакстера.
КЛЮЧОВІ СЛОВА: n-арна група, теорема Поста, комутативність, гомоморфізм, групова дія, рівняння Янга-Бакстера

One of the most promising directions in generalizing physical theories is the consideration of higher arity algebras [1], in other words ternary and n-ary algebras, in which the binary composition law is substituted by a ternary or n-ary one [2].

Firstly, ternary algebraic operations (with the arity $n = 3$) were introduced already in the XIX-th century by A. Cayley in 1845 and later by J. J. Silvester in 1883. The notion of an n-ary group was introduced in 1928 by [3] (inspired by E. Nöther) and is a natural generalization of the notion of a group. Even before this, in 1924, a particular case, that is, the ternary group of idempotents, was used in [4] to study infinite abelian groups. The important coset theorem of Post explained the connection between n-ary groups and their covering binary groups [5]. The next step in study of n-ary groups was the Gluskin-Hosszú theorem [6, 7]. Another definition of n-ary groups can be given as a universal algebra with additional laws [8] or identities containing special elements [9].

The representation theory of (binary) groups [10, 11] plays an important role in their physical applications [12]. It is initially based on a matrix realization of the group elements with the abstract group action realized as the usual matrix multiplication [13, 14]. The cubic and n-ary generalizations of matrices and determinants were made in [15, 16], and their physical application appeared in [17, 18]. In general, particular questions of n-ary group representations were considered, and matrix representations derived, by the author [19], and some general theorems connecting representations of binary and n-ary groups were presented in [20]. The intention here is to generalize the above constructions of n-ary group representations to more complicated and nontrivial cases.

In physics, the most applicable structures are the nonassociative Grassmann, Clifford and Lie algebras [21–23], and so their higher arity generalizations play the key role in further applications. Indeed, the ternary analog of Clifford algebra was

*On leave of absence from V.N. Karazin Kharkov National University, Svoboda Sq. 4, Kharkov 61022, Ukraine.

considered in [24], and the ternary analog of Grassmann algebra [25] was exploited to construct various ternary extensions of supersymmetry [26].

The construction of realistic physical models is based on Lie algebras, such that the fields take their values in a concrete binary Lie algebra [23]. In the higher arity studies, the standard Lie bracket is replaced by a linear n-ary bracket, and the algebraic structure of the corresponding model is defined by the additional characteristic identity for this generalized bracket, corresponding to the Jacobi identity [2]. There are two possibilities to construct the generalized Jacobi identity: 1) The Lie bracket is a derivation by itself; 2) A double Lie bracket vanishes, when antisymmetrized with respect to its entries. The first case leads to the so called Filippov algebras [27] (or n-Lie algebra) and second case corresponds to generalized Lie algebras [28] (or higher order Lie algebras).

The infinite-dimensional version of n-Lie algebras are the Nambu algebras [29, 30], and their n-bracket is given by the Jacobian determinant of n functions, the Nambu bracket, which in fact satisfies the Filippov identity [27]. Recently, the ternary Filippov algebras were successfully applied to a three-dimensional superconformal gauge theory describing the effective worldvolume theory of coincident $M2$-branes of M-theory [31–33]. The infinite-dimensional Nambu bracket realization [34] gave the possibility to describe a condensate of nearly coincident $M2$-branes [35].

From another side, Hopf algebras [36–38] play a fundamental role in quantum group theory [39, 40]. Previously, their Von Neumann generalization was introduced in [41–43], their actions on the quantum plane were classified in [44], and ternary Hopf algebras were defined and studied in [45, 46].

The goal of this paper is to give a comprehensive review of polyadic systems and their representations. First, we classify general polyadic systems and introduce n-ary semigroups and groups. Then we consider their homomorphisms and multiplace generalizations, paying attention to their associativity. We define multiplace representations and multiactions, and give examples of matrix representations for some ternary groups. We define and investigate ternary algebras and Hopf algebras, study their properties and give some examples. At the end we consider some ternary generalizations of quantum groups and the Yang-Baxter equation.

PRELIMINARIES

Let G be a non-empty set (underlying set, universe, carrier), its elements we denote by lower-case Latin letters $g_i \in G$. The `n-tuple` (or `polyad`) $g_1, \ldots, g_n$ of elements from G is denoted by $(g_1, \ldots, g_n)$. The Cartesian product[1] $\overbrace{G \times \ldots \times G}^{n} = G^{\times n}$ consists of all n-tuples $(g_1, \ldots, g_n)$, such that $g_i \in G$, $i = 1, \ldots, n$. For all equal elements $g \in G$, we denote n-tuple (polyad) by power (g^n). If the number of elements in the n-tuple is clear from the context or is not important, we denote it with one bold letter $(\boldsymbol{g})$, in other cases we use the power in brackets $\left(\boldsymbol{g}^{(n)}\right)$. We now introduce two important constructions on sets.

The `i-projection` of the Cartesian product $G^{\times n}$ on its i-th "axis" is the map $\mathsf{Pr}_i^{(n)} : G^{\times n} \to G$ such that $(g_1, \ldots g_i, \ldots, g_n) \longmapsto g_i$.

The `i-diagonal` $\mathsf{Diag}_n : G \to G^{\times n}$ sends one element to the equal element n-tuple $g \longmapsto (g^n)$.

The one-point set $\{\bullet\}$ can be treated as a unit for the Cartesian product, since there are bijections between G and $G \times \{\bullet\}^{\times n}$, where G can be on any place. On the Cartesian product $G^{\times n}$ one can define a polyadic (n-ary, n-adic, if it is necessary to specify n, its arity or rank) operation $\mu_n : G^{\times n} \to G$. For operations we use small Greek letters and place arguments in square brackets $\mu_n [\boldsymbol{g}]$. The operations with $n = 1, 2, 3$ are called `unary, binary and ternary`. The case $n = 0$ is special and corresponds to fixing a distinguished element of G, a "constant" $c \in G$, and it is called a `0-ary operation` $\mu_0^{(c)}$, which maps the one-point set $\{\bullet\}$ to G, such that $\mu_0^{(c)} : \{\bullet\} \to G$, and formally has the value $\mu_0^{(c)} [\{\bullet\}] = c \in G$. The 0-ary operation "kills" arity, which can be seen from the following [47]: the composition of n-ary and m-ary operations $\mu_n \circ \mu_m$ gives $(n + m - 1)$-ary operation by

$$\mu_{n+m-1} [\boldsymbol{g}, \boldsymbol{h}] = \mu_n [\boldsymbol{g}, \mu_m [\boldsymbol{h}]] . \tag{1}$$

Then, if to compose μ_n with the 0-ary operation $\mu_0^{(c)}$, we obtain

$$\mu_{n-1}^{(c)} [\boldsymbol{g}] = \mu_n [\boldsymbol{g}, c] , \tag{2}$$

because $\boldsymbol{g}$ is a polyad of length $(n - 1)$. So, it is necessary to make a clear distinction between the 0-ary operation $\mu_0^{(c)}$ and its value c in G, as will be seen and will become important below.

A `polyadic system` G is a set G which is closed under polyadic operations.

We will write G $= \langle \text{set} \,|\, \text{operations} \rangle$ or G $= \langle \text{set} \,|\, \text{operations} \,|\, \text{relations} \rangle$, where "relations" are some additional properties of operations (e.g., associativity conditions for semigroups or cancellation properties). In such a definition it is not necessary to list the images of 0-ary operations (e.g. the unit or zero in groups), as is done in various other definitions. Here, we mostly consider concrete polyadic systems with one "chief" (fundamental) n-ary operation μ_n, which is called `polyadic multiplication` (or `n-ary multiplication`).

[1] We place the sign for the Cartesian product $(\times)$ into the power, because the same abbreviation will also be used below for other types of product.

A n-ary system $\mathrm{G}_n = \langle G \mid \mu_n \rangle$ is a set G closed under one n-ary operation μ_n (without any other additional structure).

Note that a set with one closed binary operation without any other relations was called a groupoid by Hausmann and Ore [48] (see, also [49]). However, nowadays the term "groupoid" is widely used in category theory and homotopy theory for a different construction with binary multiplication, the so-called Brandt groupoid [50] (see, also, [51]). Alternatively, and much later on, Bourbaki [52] introduced the term "magma" for binary systems. Then, the above terms were extended to the case of one fundamental n-ary operation as well. Nevertheless, we will use some neutral notations "polyadic system" and "n-ary system" (when arity n is fixed/known/important), which adequately indicates all of their main properties.

Let us consider the changing arity problem:

For a given n-ary system $\langle G \mid \mu_n \rangle$ to construct another polyadic system $\langle G \mid \mu'_{n'} \rangle$ over the same set G, which has multiplication with a different arity n'.

The formulas (1) and (2) give us the simplest examples of how to change the arity of a polyadic system. In general, there are 3 ways:

1. *Iterating.* Using composition of the operation μ_n with itself, one can increase the arity from n to n'_{iter} (as in (1)) without changing the signature of the system. We denote the number of iterating multiplications by ℓ_μ, and use the bold Greek letters $\boldsymbol{\mu}_n^{\ell_\mu}$ for the resulting composition of n-ary multiplications, such that

$$\mu'_{n'} = \boldsymbol{\mu}_n^{\ell_\mu} \stackrel{def}{=} \overbrace{\mu_n \circ \left(\mu_n \circ \ldots \left(\mu_n \times \mathrm{id}^{\times(n-1)}\right) \ldots \times \mathrm{id}^{\times(n-1)}\right)}^{\ell_\mu}, \tag{3}$$

where

$$n' = n_{iter} = \ell_\mu (n-1) + 1, \tag{4}$$

which gives the length of a polyad $(\boldsymbol{g})$ in the notation $\boldsymbol{\mu}_n^{\ell_\mu}[\boldsymbol{g}]$. Without assuming associativity there many variants for placing μ_n's among id's in the r.h.s. of (3). The operation $\boldsymbol{\mu}_n^{\ell_\mu}$ is named a long product [3] or derived [53].

2. *Reducing (Collapsing).* Using n_c distinguished elements or constants (or n_c additional 0-ary operations $\mu_0^{(c_i)}$, $i = 1, \ldots n_c$), one can decrease arity from n to n'_{red} (as in (2)), such that[2]

$$\mu'_{n'} = \mu_{n'}^{(c_1 \ldots c_{n_c})} \stackrel{def}{=} \mu_n \circ \left(\overbrace{\mu_0^{(c_1)} \times \ldots \times \mu_0^{(c_{n_c})}}^{n_c} \times \mathrm{id}^{\times(n-n_c)} \right), \tag{5}$$

where

$$n' = n_{red} = n - n_c, \tag{6}$$

and the 0-ary operations $\mu_0^{(c_i)}$ can be on any places.

3. *Mixing.* Changing (increasing or decreasing) arity may be done by combining iterating and reducing (maybe with additional operations of different arity). If we do not use additional operations, the final arity can be presented in a general form using (4) and (6). It will depend on the order of iterating and reducing, and so we have two subcases:

   (a) *Iterating→Reducing.* We have
   $$n' = n_{iter\to red} = \ell_\mu (n-1) - n_c + 1. \tag{7}$$
   The maximal number of constants (when $n'_{iter\to red} = 2$) is equal to
   $$n_c^{\max} = \ell_\mu (n-1) - 1 \tag{8}$$
   and can be increased by increasing the number of multiplications ℓ_μ.

   (b) *Reducing→Iterating.* We obtain
   $$n' = n_{red\to iter} = \ell_\mu (n - 1 - n_c) + 1. \tag{9}$$
   Now the maximal number of constants is
   $$n_c^{\max} = n - 2 \tag{10}$$
   and this is achieved only when $\ell_\mu = 1$.

[2] In [54] $\mu_n^{(c_1 \ldots c_{n_c})}$ is named a retract (which term is already busy and widely used in category theory for another construction).

To give examples of the third (mixed) case we put $n = 4$, $\ell_\mu = 3$, $n_c = 2$ for both subcases of opposite ordering:

1. *Iterating→Reducing.* We can put

$$\mu_8^{(c_1,c_2)\prime}\left[\boldsymbol{g}^{(8)}\right] = \mu_4\left[g_1, g_2, g_3, \mu_4\left[g_4, g_5, g_6, \mu_4\left[g_7, g_8, c_1, c_2\right]\right]\right]. \tag{11}$$

2. *Reducing→Iterating.* We can have

$$\mu_4^{(c_1,c_2)\prime}\left[\boldsymbol{g}^{(4)}\right] = \mu_4\left[g_1, c_1, c_2, \mu_4\left[g_2, c_1, c_2, \mu_4\left[g_3, c_1, c_2, g_4\right]\right]\right]. \tag{12}$$

It is important to find conditions where iterating and reducing compensate each other, i.e. they do not change arity overall. Indeed, let the number of the iterating multiplications ℓ_μ be fixed, then we can find such a number of reducing constants $n_c^{(0)}$, such that the final arity will coincide with the initial arity n. The result will depend on the order of operations. There are two cases:

1. *Iterating→Reducing.* For the number of reducing constants $n_c^{(0)}$ we obtain from (4) and (6)

$$n_c^{(0)} = (n-1)(\ell_\mu - 1), \tag{13}$$

such that there is no restriction on ℓ_μ.

2. *Reducing→Iterating.* For $n_c^{(0)}$ we get

$$n_c^{(0)} = \frac{(n-1)(\ell_\mu - 1)}{\ell_\mu}, \tag{14}$$

and now $\ell_\mu \leq n-1$. The requirement that $n_c^{(0)}$ should be an integer gives two further possibilities

$$n_c^{(0)} = \begin{cases} \dfrac{n-1}{2}, & \ell_\mu = 2, \\ n-2, & \ell_\mu = n-1. \end{cases} \tag{15}$$

The above relations can be useful in the study of various n-ary multiplication structures and their presentation in special form is needed in concrete problems.

SPECIAL ELEMENTS AND PROPERTIES OF POLYADIC SYSTEMS

Let us recall the definitions of some standard algebraic systems and their special elements, which will be considered in this paper, using our notation.

A `zero` of a polyadic system is a distinguished element z (and the corresponding 0-ary operation $\mu_0^{(z)}$) such that for any $(n-1)$-tuple (polyad) $\boldsymbol{g} \in G^{\times(n-1)}$ we have

$$\mu_n[\boldsymbol{g}, z] = z, \tag{16}$$

where z can be on any place in the l.h.s. of (16).

There is only one zero (if its place is not fixed) which can be possible in a polyadic system. As in the binary case, an analog of positive powers of an element [5] should coincide with the number of multiplications ℓ_μ in the iterating (3).

A (positive) `polyadic power` of an element is

$$g^{\langle \ell_\mu \rangle} = \boldsymbol{\mu}_n^{\ell_\mu}\left[g^{\ell_\mu(n-1)+1}\right]. \tag{17}$$

An element of a polyadic system g is called ℓ_μ-`nilpotent` (or simply `nilpotent` for $\ell_\mu = 1$), if there exist such ℓ_μ that

$$g^{\langle \ell_\mu \rangle} = z. \tag{18}$$

A polyadic system with zero z is called ℓ_μ-`nilpotent`, if there exists ℓ_μ such that for any $(\ell_\mu(n-1)+1)$-tuple (polyad) $\boldsymbol{g}$ we have

$$\boldsymbol{\mu}_n^{\ell_\mu}[\boldsymbol{g}] = z. \tag{19}$$

Therefore, the `index of nilpotency` (number of elements whose product is zero) of an ℓ_μ-nilpotent n-ary system is $(\ell_\mu(n-1)+1)$, while its polyadic power is ℓ_μ.

A `polyadic (n-ary) identity` (or neutral element) of a polyadic system is a distinguished element e (and the corresponding 0-ary operation $\mu_0^{(e)}$) such that for any element $g \in G$ we have

$$\mu_n\left[g, e^{n-1}\right] = g, \tag{20}$$

where g can be on any place in the l.h.s. of (20).

In binary groups the identity is the only neutral element, while in polyadic systems, there exist `neutral polyads` $\boldsymbol{n}$ consisting of elements of G satisfying

$$\mu_n\left[g, \boldsymbol{n}\right] = g, \tag{21}$$

where g can be also on any place. The neutral polyads are not determined uniquely. It follows from (20) that the sequence of polyadic identities e^{n-1} is a neutral polyad.

An element of a polyadic system g is called ℓ_μ-`idempotent` (or simply `idempotent` for $\ell_\mu = 1$), if there exist such ℓ_μ that

$$g^{\langle \ell_\mu \rangle} = g. \tag{22}$$

Both zero and the identity are ℓ_μ-idempotents with arbitrary ℓ_μ. We define `(total) associativity` as the invariance of the composition of two n-ary multiplications

$$\boldsymbol{\mu}_n^2\left[\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{u}\right] = \mu_n\left[\boldsymbol{g}, \mu_n\left[\boldsymbol{h}\right], \boldsymbol{u}\right] = inv \tag{23}$$

under placement of the internal multiplication in r.h.s. with a fixed order of elements in the whole polyad of $(2n-1)$ elements $\boldsymbol{t}^{(2n-1)} = (\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{u})$. Informally, “internal brackets/multiplication can be moved on any place”, which gives n relations

$$\mu_n \circ \left(\mu_n \times \mathrm{id}^{\times(n-1)}\right) = \ldots = \mu_n \circ \left(\mathrm{id}^{\times(n-1)} \times \mu_n\right). \tag{24}$$

There are many other particular kinds of associativity which were introduced in [55] and studied in [56, 57]. Here we will confine ourselves the most general, total associativity (23). In this case, the iteration does not depend on the placement of internal multiplications in the r.h.s of (3).

A `polyadic semigroup` (n-`ary semigroup`) is a n-ary system in which the operation is associative, or $\mathrm{G}_n^{semigrp} = \langle G \mid \mu_n \mid \text{associativity} \rangle$.

In a polyadic system with zero (16) one can have `trivial associativity`, when all n terms are (23) are equal to zero, i.e.

$$\boldsymbol{\mu}_n^2\left[\boldsymbol{g}\right] = z \tag{25}$$

for any $(2n-1)$-tuple $\boldsymbol{g}$. Therefore, we state that

Any 2-nilpotent n-ary system (having index of nilpotency $(2n-1)$) is a polyadic semigroup.

In the case of changing arity one should use in (25) not the changed final arity n', but the “real” arity which is n for the reducing case and $\ell_\mu(n-1)+1$ for all other cases. Let us give some examples.

In the mixed (interacting-reducing) case with $n = 2$, $\ell_\mu = 3$, $n_c = 1$, we have a ternary system $\langle G \mid \mu_3 \rangle$ iterated from a binary system $\left\langle G \mid \mu_2, \mu_0^{(c)} \right\rangle$ with one distinguished element c (or an additional 0-ary operation)[3]

$$\mu_3^{(c)}\left[g, h, u\right] = \left(g \cdot \left(h \cdot \left(u \cdot c\right)\right)\right), \tag{26}$$

where for binary multiplication we denote $g \cdot h = \mu_2\left[g, h\right]$. Thus, if the ternary system $\left\langle G \mid \mu_3^{(c)} \right\rangle$ is nilpotent of index 7 (see 9), then it is a ternary semigroup (because $\mu_3^{(c)}$ is trivially associative) independently of the associativity of μ_2 (see, e.g. [19]).

It is very important to find the `associativity preserving` conditions (constructions), where an associative initial operation μ_n leads to an associative final operation $\mu'_{n'}$ during the change of arity.

An associativity preserving reduction can be given by the construction of a binary associative operation using $(n-2)$-tuple $\boldsymbol{c}$ consisting of $n_c = n-2$ different constants

$$\mu_2^{(\boldsymbol{c})}\left[g, h\right] = \mu_n\left[g, \boldsymbol{c}, h\right]. \tag{27}$$

Associativity preserving mixing constructions with different arities and places were considered in [54, 57, 58].

An associative polyadic system with identity (20) is called a `polyadic monoid`.

The structure of any polyadic monoid is fixed [59]: it can be obtained by iterating a binary operation [60] (for polyadic groups this was shown in [3]).

In polyadic systems, there are several analogs of binary commutativity. The most straightforward one comes from commutation of the multiplication with permutations.

[3]This construction is named the b-derived groupoid in [54].

A polyadic system is σ-`commutative`, if $\mu_n = \mu_n \circ \sigma$, or

$$\mu_n [\boldsymbol{g}] = \mu_n [\sigma \circ \boldsymbol{g}], \tag{28}$$

where $\sigma \circ \boldsymbol{g} = \left(g_{\sigma(1)}, \ldots, g_{\sigma(n)}\right)$ is a permutated polyad and σ is a fixed element of S_n, the permutation group on n elements. If (28) holds for all $\sigma \in S_n$, then a polyadic system is `commutative`.

A special type of the σ-commutativity

$$\mu_n [g, \boldsymbol{t}, h] = \mu_n [h, \boldsymbol{t}, g], \tag{29}$$

where $\boldsymbol{t}$ is any fixed $(n-2)$-polyad, is called `semicommutativity`. So for a n-ary semicommutative system we have

$$\mu_n \left[g, h^{n-1}\right] = \mu_n \left[h^{n-1}, g\right]. \tag{30}$$

If a n-ary semigroup $\mathrm{G}^{semigrp}$ is iterated from a commutative binary semigroup with identity, then $\mathrm{G}^{semigrp}$ is semicommutative.

Let G be the set of natural numbers $\mathbb{N}$, and the 5-ary multiplication is defined by

$$\mu_5 [\boldsymbol{g}] = g_1 - g_2 + g_3 - g_4 + g_5, \tag{31}$$

then $\mathrm{G}_5^{\mathbb{N}} = \langle \mathbb{N}, \mu_5 \rangle$ is a semicommutative 5-ary monoid having the identity $e_g = \mu_5 \left[g^5\right] = g$ for each $g \in \mathbb{N}$. Therefore, $\mathrm{G}_5^{\mathbb{N}}$ is the idempotent monoid.

Another possibility is to generalize the binary `mediality` in semigroups

$$(g_{11} \cdot g_{12}) \cdot (g_{21} \cdot g_{22}) = (g_{11} \cdot g_{21}) \cdot (g_{12} \cdot g_{22}), \tag{32}$$

which, obviously, follows from binary commutativity. But for n-ary systems they are different. It is seen that the mediality should contain $(n+1)$ multiplications, it is a relation between $n \times n$ elements, and therefore can be presented in a matrix from. The latter can be achieved by placing the arguments of the external multiplication in a column.

A polyadic system is `medial` (or `entropic`), if [56, 61]

$$\mu_n \begin{bmatrix} \mu_n [g_{11}, \ldots, g_{1n}] \\ \vdots \\ \mu_n [g_{n1}, \ldots, g_{nn}] \end{bmatrix} = \mu_n \begin{bmatrix} \mu_n [g_{11}, \ldots, g_{n1}] \\ \vdots \\ \mu_n [g_{1n}, \ldots, g_{nn}] \end{bmatrix}. \tag{33}$$

For polyadic semigroups we use the notation (3) and can present the mediality as follows

$$\boldsymbol{\mu}_n^n [\boldsymbol{G}] = \boldsymbol{\mu}_n^n \left[\boldsymbol{G}^T\right], \tag{34}$$

where $\boldsymbol{G} = \|g_{ij}\|$ is the $n \times n$ matrix of elements and $\boldsymbol{G}^T$ is its transpose. The semicommutative polyadic semigroups are medial, as in the binary case, but, in general (except $n = 3$) not vice versa [62]. A more general concept is σ-permutability [63], such that the mediality is its particular case with $\sigma = (1, n)$.

A polyadic system is `cancellative`, if

$$\mu_n [g, \boldsymbol{t}] = \mu_n [h, \boldsymbol{t}] \Longrightarrow g = h, \tag{35}$$

where g, h can be on any place. This means that the mapping μ_n is one-to-one in each variable. If g, h are on the same i-th place on both sides, the polyadic system is called i-`cancellative`.

The `left` and `right` cancellativity are 1-cancellativity and n-cancellativity respectively. A right and left cancellative n-ary semigroup is `cancellative` (with respect to the same subset).

A polyadic system is called (uniquely) i-`solvable`, if for all polyads $\boldsymbol{t}$, $\boldsymbol{u}$ and element h, one can (uniquely) resolve the equation (with respect to h) for the fundamental operation

$$\mu_n [\boldsymbol{u}, h, \boldsymbol{t}] = g \tag{36}$$

where h can be on any i-th place.

A polyadic system which is uniquely i-solvable for *all* places i is called a n-`ary` (or `polyadic`) `quasigroup`.

It follows, that, if (36) uniquely i-solvable for all places, than

$$\boldsymbol{\mu}_n^{\ell_\mu} [\boldsymbol{u}, h, \boldsymbol{t}] = g \tag{37}$$

can be (uniquely) resolved with respect to h on any place.

An associative polyadic quasigroup is called a n-`ary` (or `polyadic`) `group`.

The above definition is the most general one, but it is overdetermined. Much work on polyadic groups was done [64] to minimize the set of axioms (solvability not in all places [5, 65], decreasing or increasing the number of unknowns in determining equations [66]) or construction in terms of additionally defined objects (various analogs of the identity and sequences [67]), as well as using not total associativity, but instead various partial ones [57, 68, 69].

In a polyadic group the only solution of (36) is called a querelement of g and denoted by $\bar{g}$ [3], such that

$$\mu_n [\boldsymbol{h}, \bar{g}] = g, \tag{38}$$

where $\bar{g}$ can be on any place. So, any idempotent g coincides with its querelement $\bar{g} = g$. It follows from (38) and (21), that the polyad

$$\boldsymbol{n}_g = \left(g^{n-2}\bar{g}\right) \tag{39}$$

is neutral for any element of a polyadic group, where $\bar{g}$ can be on any place. If this i-th place is important, then we write $\boldsymbol{n}_{g;i}$. The number of relations in (38) can be reduced from n (the number of possible places) to only 2 (when g is on the first and last places [3, 70], or on some other 2 places). In a polyadic group the Dörnte relations

$$\mu_n [g, \boldsymbol{n}_{h;i}] = \mu_n [\boldsymbol{n}_{h;j}, g] = g \tag{40}$$

hold true for any allowable i, j. In the case of a binary group the relations (40) become $g \cdot h \cdot h^{-1} = h \cdot h^{-1} \cdot g = g$.

The relation (38) can be treated as a definition of the unary queroperation

$$\bar{\mu}_1 [g] = \bar{g}. \tag{41}$$

A polyadic group is a universal algebra

$$\mathrm{G}_n^{grp} = \langle G \mid \mu_n, \bar{\mu}_1 \mid \text{associativity, Dörnte relations} \rangle , \tag{42}$$

where μ_n is a n-ary associative operation and $\bar{\mu}_1$ is the queroperation.

A straightforward generalization of the queroperation concept and corresponding definitions can be made by substituting in the above formulas (38)–(41) the n-ary multiplication μ_n by iterating the multiplication $\boldsymbol{\mu}_n^{\ell_\mu}$ (3) (cf. [71] for $\ell_\mu = 2$).

Let us define the querpower k of g recursively

$$\bar{g}^{\langle\langle k\rangle\rangle} = \overline{\left(\bar{g}^{\langle\langle k-1\rangle\rangle}\right)}, \tag{43}$$

where $\bar{g}^{\langle\langle 0\rangle\rangle} = g$, $\bar{g}^{\langle\langle 1\rangle\rangle} = \bar{g}$, or as the k composition $\bar{\mu}_1^{\circ k} = \overbrace{\bar{\mu}_1 \circ \bar{\mu}_1 \circ \ldots \circ \bar{\mu}_1}^{k}$ of the queroperation (41).

For instance [66], $\bar{\mu}_1^{\circ 2} = \boldsymbol{\mu}_n^{n-3}$, such that for any ternary group $\bar{\mu}_1^{\circ 2} = \mathrm{id}$, i.e. one has $\bar{\bar{g}} = g$. Using the queroperation in polyadic groups we can define the negative polyadic power of an element g by the following recursive relation

$$\mu_n \left[g^{\langle \ell_\mu - 1\rangle}, g^{n-2}, g^{\langle -\ell_\mu\rangle}\right] = g, \tag{44}$$

or (after use of (17)) as a solution of the equation

$$\boldsymbol{\mu}_n^{\ell_\mu} \left[g^{\ell_\mu (n-1)}, g^{\langle -\ell_\mu\rangle}\right] = g. \tag{45}$$

It is known that the querpower and the polyadic power are mutually connected [72]. Here, we reformulate this connection using the so called Heine numbers [73] or q-deformed numbers [74]

$$[[k]]_q = \frac{q^k - 1}{q - 1}, \tag{46}$$

which have the “nondeformed” limit $q \to 1$ as $[k]_q \to k$. Then

$$\bar{g}^{\langle\langle k\rangle\rangle} = g^{\left\langle -[[k]]_{2-n}\right\rangle}, \tag{47}$$

which can be treated as follows: the querpower coincides with the negative polyadic deformed power with a “deformation” parameter q which is equal to the “deviation” $(2 - n)$ from the binary group.

HOMOMORPHISMS OF POLYADIC SYSTEMS

Let $\mathrm{G}_n = \langle G; \mu_n \rangle$ and $\mathrm{G}'_{n'} = \langle G'; \mu'_{n'} \rangle$ be two polyadic systems of any kind (quasigroup, semigroup, group, etc.). If they have the multiplications of the same arity $n = n'$, then one can define the mappings from G_n to G'_n. Usually such polyadic systems are `similar`, and we call mappings between them `the equiary mappings`.

Let us take $n+1$ mappings $\varphi_i^{GG'} : G \to G'$, $i = 1, \ldots, n+1$. An ordered system of mappings $\left\{ \varphi_i^{GG'} \right\}$ is called a `homotopy` from G_n to G'_n, if

$$\varphi_{n+1}^{GG'} \left(\mu_n \left[g_1, \ldots, g_n \right] \right) = \mu'_n \left[\varphi_1^{GG'} \left(g_1 \right), \ldots, \varphi_n^{GG'} \left(g_n \right) \right], \quad g_i \in G. \tag{48}$$

In general, one should add to this definition the “mapping” of the multiplications

$$\mu_n \overset{\psi_{nn'}^{(\mu\mu')}}{\mapsto} \mu'_{n'}. \tag{49}$$

In such a way, homotopy can be defined as the extended system of mappings $\left\{ \varphi_i^{GG'}; \psi_{nn}^{(\mu\mu')} \right\}$.

The existence of the additional “mapping” $\psi_{nn}^{(\mu\mu')}$ acting on the second component of $\langle G; \mu_n \rangle$ is tacitly implied. We will write/mention the “mappings” $\psi_{nn'}^{(\mu\mu')}$ manifestly, e.g.,

$$\mathrm{G}_n \overset{\left\{ \varphi_i^{GG'}; \psi_{nn}^{(\mu\mu')} \right\}}{\rightrightarrows} \mathrm{G}'_{n'}, \tag{50}$$

only as needed. If all the components $\varphi_i^{GG'}$ of a homotopy are bijections, it is called an `isotopy`. In case of polyadic quasigroups [56] all mappings $\varphi_i^{GG'}$ are usually taken as permutations of the same underlying set $G = G'$. If the multiplications are also coincide $\mu_n = \mu'_n$, then $\left\{ \varphi_i^{GG}; \mathrm{id} \right\}$ is called an `autotopy` of the polyadic system G_n. Various properties of homotopy in universal algebras were studied, e.g. in [75, 76].

A `homomorphism` from G_n to G'_n is given, if there exists a mapping $\varphi^{GG'} : G \to G'$ satisfying

$$\varphi^{GG'} \left(\mu_n \left[g_1, \ldots, g_n \right] \right) = \mu'_n \left[\varphi^{GG'} \left(g_1 \right), \ldots, \varphi^{GG'} \left(g_n \right) \right], \quad g_i \in G. \tag{51}$$

Usually the homomorphism is denoted by the same one letter $\varphi^{GG'}$, while it would be more consistent to use for its notation the extended `pair` of mappings $\left\{ \varphi^{GG'}; \psi_{nn}^{(\mu\mu')} \right\}$. We will use both notations on a par.

We first mention a small subset of known generalizations of the homomorphism (for bibliography till 1982 see, e.g., [77]) and then introduce a concrete construction for an analogous mapping which can change the arity of the multiplication (fundamental operation) without introducing additional (term) operations. A general approach to mappings between free algebraic systems was initiated in [78], where the so-called basic mapping formulas for generators were introduced, and its generalization to many-sorted algebras was given in [79]. In [80] it was shown that the construction of all homomorphisms between similar polyadic systems can be reduced to some homomorphisms between corresponding mono-unary algebras [81]. The notion of n-ary homomorphism is realized as a sequence of n consequent homomorphisms φ_i, $i = 1, \ldots, n$, of n similar polyadic systems

$$\overbrace{\mathrm{G}_n \overset{\varphi_1}{\to} \mathrm{G}'_n \overset{\varphi_2}{\to} \ldots \overset{\varphi_{n-1}}{\to} \mathrm{G}''_n \overset{\varphi_n}{\to} \mathrm{G}'''_n}^{n} \tag{52}$$

(generalizing Post’s n-adic substitutions [5]) was introduced in [82], and studied in [83, 84].

The above constructions do not change the arity of polyadic systems, because they are based on the corresponding diagram which gives a definition of an `equiary` mapping. To change arity one has to:

1) add another equiary diagram with `additional` operations using `the same` formula (51), where both do not change arity;

2) use `one` modified (and not equiary) diagram and the underlying formula (51) by themselves, which will allow us to change arity `without` introducing additional operations.

The first way leads to the concept of `weak homomorphism` which was introduced in [85–87] for non-indexed algebras and in [88] for indexed algebras, then developed in [89] for Boolean and Post algebras, in [90] for coalgebras and F-algebras [91] (see also [92]). To define the weak homomorphism in our notation we should incorporate into the polyadic systems $\langle G; \mu_n \rangle$ and $\langle G'; \mu'_{n'} \rangle$ the following additional term operations of opposite arity $\nu_{n'} : G^{\times n'} \to G$ and $\nu'_n : G'^{\times n} \to G'$ and consider `two` equiary mappings between $\langle G; \mu_n, \nu_{n'} \rangle$ and $\langle G'; \mu'_{n'}, \nu'_n \rangle$.

A `weak homomorphism` from $\langle G; \mu_n, \nu_{n'} \rangle$ to $\langle G', \mu'_{n'}, \nu'_n \rangle$ is given, if there exists a mapping $\varphi^{GG'} : G \to G'$ satisfying `two` relations simultaneously

$$\varphi^{GG'} \left(\mu_n \left[g_1, \ldots, g_n \right] \right) = \nu'_n \left[\varphi^{GG'} \left(g_1 \right), \ldots, \varphi^{GG'} \left(g_n \right) \right], \tag{53}$$

$$\varphi^{GG'} \left(\nu_{n'} \left[g_1, \ldots, g_{n'} \right] \right) = \mu'_{n'} \left[\varphi^{GG'} \left(g_1 \right), \ldots, \varphi^{GG'} \left(g_{n'} \right) \right]. \tag{54}$$

If only one of the relations (53) or (54) holds, such a mapping is called a `semi-weak homomorphism` [93]. If $\varphi^{GG'}$ is bijective, then it defines a `weak isomorphism`. Any weak epimorphism can be decomposed into a homomorphism and a weak isomorphism [94], and therefore the study of weak homomorphisms reduces to weak isomorphisms (see also [95–97]).

MULTIPLACE MAPPINGS OF POLYADIC SYSTEMS

Let us turn to the second way of changing the arity of the multiplication and use only one relation which we then modify in some natural manner. First, recall that in any set G there always exists the additional distinguished mapping, viz. the identity id_G. We use the multiplication μ_n with its combination of id_G. We define an (ℓ_{id}-`intact) id-product` for the polyadic system $\langle G; \mu_n \rangle$ as

$$\mu_n^{(\ell_{\mathrm{id}})} = \mu_n \times \left(\mathrm{id}_G \right)^{\times \ell_{\mathrm{id}}}, \tag{55}$$

$$\mu_n^{(\ell_{\mathrm{id}})} : G^{\times (n + \ell_{\mathrm{id}})} \to G^{\times (1 + \ell_{\mathrm{id}})}. \tag{56}$$

To indicate the exact i-th place of μ_n in the r.h.s. of (55), we write $\mu_n^{(\ell_{\mathrm{id}})} (i)$, as needed. Here we use the id-product to generalize the homomorphism and consider mappings between polyadic systems of different arity. It follows from (56) that, if the image of the id-product is G alone, than $\ell_{\mathrm{id}} = 0$. Let us introduce a `multiplace mapping` $\Phi_k^{(n,n')}$ acting as follows

$$\Phi_k^{(n,n')} : G^{\times k} \to G'. \tag{57}$$

We are allowed to take only one upper $\Phi_k^{(n,n')}$, because of one G' in the upper right corner. Since we already know that the lower right corner is exactly $G'^{\times n'}$ (as a pre-image of one multiplication $\mu'_{n'}$), the lower horizontal arrow should be a product of n' multiplace mappings $\Phi_k^{(n,n')}$. So we can write a definition of a multiplace analog of homomorphisms which changes the arity of the multiplication using one relation.

A k-`place heteromorphism` from G_n to $\mathrm{G}'_{n'}$ is given, if there exists a k-place mapping $\Phi_k^{(n,n')}$ (57) such that the corresponding defining equation (a modification of (51)) depends on the place i of μ_n in (55). For $i = 1$ it can read as

$$\Phi_k^{(n,n')} \begin{pmatrix} \mu_n \left[g_1, \ldots, g_n \right] \\ g_{n+1} \\ \vdots \\ g_{n+\ell_{\mathrm{id}}} \end{pmatrix} = \mu'_{n'} \left[\Phi_k^{(n,n')} \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix}, \ldots, \Phi_k^{(n,n')} \begin{pmatrix} g_{k(n'-1)} \\ \vdots \\ g_{kn'} \end{pmatrix} \right]. \tag{58}$$

This notion is motivated by [98, 99], where mappings between objects from different categories were considered and called 'chimera morphisms'. See, also, [100].

In the particular case $n = 3$, $n' = 2$, $k = 2$, $\ell_{\mathrm{id}} = 1$ we have

$$\Phi_2^{(3,2)} \begin{pmatrix} \mu_3 \left[g_1, g_2, g_3 \right] \\ g_4 \end{pmatrix} = \mu'_2 \left[\Phi_2^{(3,2)} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \Phi_2^{(3,2)} \begin{pmatrix} g_3 \\ g_4 \end{pmatrix} \right]. \tag{59}$$

This formula was used in the construction of the bi-element representations of ternary groups [19]. Consider the example.

Let $G = M_2^{adiag} (\mathbb{K})$, a set of antidiagonal 2×2 matrices over the field $\mathbb{K}$ and $G' = \mathbb{K}$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{H}$. The ternary multiplication μ_3 is a product of 3 matrices. Obviously, μ_3 is nonderived. For the elements $g_i = \begin{pmatrix} 0 & a_i \\ b_i & 0 \end{pmatrix}$, $i = 1, 2$, we construct a 2-place mapping $G \times G \to G'$ as

$$\Phi_2^{(3,2)} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = a_1 a_2 b_1 b_2, \tag{60}$$

which satisfies (59). We can introduce a standard 1-place mapping by $\varphi (g_i) = a_i b_i$. It is important to note, that $\varphi (g_i)$ satisfies (51) for a commutative field $\mathbb{K}$ only ($= \mathbb{R}, \mathbb{C}$) becoming a homomorphism, and in this case we can have the relation *between the heteromorhism $\Phi_2^{(3,2)}$ and the standard homomorphism*

$$\Phi_2^{(3,2)}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \varphi(g_1)\cdot\varphi(g_2), \tag{61}$$

where the product $(\cdot)$ in l.h.s. is taken in $\mathbb{K}$, such that (51) and (59) coincide. For the noncommutative field $\mathbb{K}$ ($=\mathbb{Q}$ or $\mathbb{H}$) we can define the heteromorphism (60) only.

A heteromorphism is called `derived`, if it can be expressed through an ordinary (1-place) homomorphism. So, in the above example the heteromorphism is derived (by formula (61)) for a commutative field $\mathbb{K}$ and nonderived for a noncommutative one.

For arbitrary n a slightly modified construction (59) with still binary final arity, defined by $n'=2$, $k=n-1$, $\ell_{\mathrm{id}}=n-2$,

$$\Phi_{n-1}^{(n,2)}\begin{pmatrix} \mu_n[g_1,\ldots,g_{n-1},h_1] \\ h_2 \\ \vdots \\ h_{n-1} \end{pmatrix} = \mu_2'\left[\Phi_{n-1}^{(n,2)}\begin{pmatrix} g_1 \\ \vdots \\ g_{n-1} \end{pmatrix}, \Phi_{n-1}^{(n,2)}\begin{pmatrix} h_1 \\ \vdots \\ h_{n-1} \end{pmatrix}\right]. \tag{62}$$

was used in [53] to study representations of n-ary groups. However, no new results compared with [19] (other than changing 3 to n in some formulas) were obtained. This reflects the fact that a major role is played by the final arity n' and the number of n-ary multiplications in the l.h.s. of (59) and (62). In the above cases, the latter number was one, but can make it arbitrary below n.

A heteromorphism is called a ℓ_μ`-ple heteromorphism`, if it contains ℓ_μ multiplications in the argument of $\Phi_k^{(n,n')}$ in its defining relation. According this definition the mapping defined by (58) is the 1_μ-ple heteromorphism. So by analogy with (55)–(56) we define a ℓ_μ`-ple` ℓ_{id}`-intact id-product` for the polyadic system $\langle G;\mu_n\rangle$ as

$$\mu_n^{(\ell_\mu,\ell_{\mathrm{id}})} = (\mu_n)^{\times\ell_\mu}\times(\mathrm{id}_G)^{\times\ell_{\mathrm{id}}}, \tag{63}$$
$$\mu_n^{(\ell_\mu,\ell_{\mathrm{id}})} : G^{\times(n\ell_\mu+\ell_{\mathrm{id}})}\to G^{\times(\ell_\mu+\ell_{\mathrm{id}})}. \tag{64}$$

A ℓ_μ`-ple` k`-place heteromorphism` from G_n to $\mathrm{G}'_{n'}$ is given, if there exists a k-place mapping $\Phi_k^{(n,n')}$ (57). The corresponding `main heteromorphism equation` is

$$\Phi_k^{(n,n')}\begin{pmatrix} \left.\begin{matrix}\mu_n[g_1,\ldots,g_n], \\ \vdots \\ \mu_n\left[g_{n(\ell_\mu-1)},\ldots,g_{n\ell_\mu}\right]\end{matrix}\right\}\ell_\mu \\ \left.\begin{matrix} g_{n\ell_\mu+1}, \\ \vdots \\ g_{n\ell_\mu+\ell_{\mathrm{id}}}\end{matrix}\right\}\ell_{\mathrm{id}} \end{pmatrix} = \mu'_{n'}\left[\Phi_k^{(n,n')}\begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix},\ldots,\Phi_k^{(n,n')}\begin{pmatrix} g_{k(n'-1)} \\ \vdots \\ g_{kn'} \end{pmatrix}\right]. \tag{65}$$

Obviously, we can consider various permutations of the multiplications on both sides, as further additional demands (associativity, commutativity, etc.), are introduced, which will be considered below. The system of equations connecting initial and final arities is

$$kn' = n\ell_\mu+\ell_{\mathrm{id}}, \tag{66}$$
$$k = \ell_\mu+\ell_{\mathrm{id}}. \tag{67}$$

Excluding ℓ_μ or ℓ_{id}, we obtain two `arity changing formulas`, respectively

$$n' = n-\frac{n-1}{k}\ell_{\mathrm{id}}, \tag{68}$$
$$n' = \frac{n-1}{k}\ell_\mu+1, \tag{69}$$

where $\frac{n-1}{k}\ell_{\mathrm{id}}\geq 1$ and $\frac{n-1}{k}\ell_\mu\geq 1$ are integer. The following inequalities hold valid

$$1\leq\ell_\mu\leq k, \tag{70}$$
$$0\leq\ell_{\mathrm{id}}\leq k-1, \tag{71}$$
$$\ell_\mu\leq k\leq(n-1)\,\ell_\mu, \tag{72}$$
$$2\leq n'\leq n, \tag{73}$$

which are important for the further classification of heteromorphisms. The main statement follows from (73):

The heteromorphism $\Phi_k^{(n,n')}$ *defined by the relation (65) decreases arity of the multiplication.*

Another important observation is the fact that only the id-product (63) with $\ell_{\text{id}} \neq 0$ leads to a change of the arity. In the extreme case, when k approaches its minimum, $k = k_{\min} = \ell_\mu$, the final arity approaches its maximum $n'_{\max} = n$, and the id-product becomes a product of ℓ_μ initial multiplications μ_n without id's, since now $\ell_{\text{id}} = 0$ in (65). Therefore, we call a heteromorphism defined by (65) with $\ell_{\text{id}} = 0$ a $k\,(=\ell_\mu)$-`place homomorphism`. The ordinary homomorphism (48) corresponds to $k = \ell_\mu = 1$, and so it is really a 1-place homomorphism. An opposite extreme case, when the final arity approaches its minimum $n'_{\min} = 2$ (the final operation is binary), corresponds to the maximal value of k, that is $k = k_{\max} = (n-1)\,\ell_\mu$. The number of id's now is $\ell_{\text{id}} = (n-2)\,\ell_\mu \geq 0$, which vanishes, when the initial operation is binary as well. This is the case of the ordinary homomorphism (48) for both binary operations $n' = n = 2$ and $k = \ell_\mu = 1$. We conclude that:

Any polyadic system can be mapped into a binary system by means of the special k-place ℓ_μ-ple heteromorphism $\Phi_k^{(n,n')}$, where $k = (n-1)\,\ell_\mu$ (we call it a `binarizing heteromorphism`) which is defined by (65) with $\ell_{\text{id}} = (n-2)\,\ell_\mu$.

In relation to the Gluskin-Hosszú theorem [7] (any n-ary group can be constructed from the special binary group and its homomorphism) our statement can be treated as:

Theorem 27. *Any n-ary system can be mapped into a binary system, using a suitable binarizing heteromorphism* $\Phi_k^{(n,2)}$ *(65).*

The case of 1-ple binarizing heteromorphism ($\ell_\mu = 1$) corresponds to the formula (62). Further requirements (associativity, commutativity, etc.) will give additional relations between multiplications and $\Phi_k^{(n,n')}$, and fix the exact structure of (65). Thus, we arrive to the following

Classification of ℓ_μ-ple heteromorphisms:

1. $n' = n'_{\max} = n \Longrightarrow \Phi_k^{(n,n)}$ *is the ℓ_μ-place or* `multiplace homomorphism`*, i.e.,*

$$k = k_{\min} = \ell_\mu. \tag{74}$$

2. $2 < n' < n \Longrightarrow \Phi_k^{(n,n')}$ *is the* `intermediate heteromorphism` *with*

$$k = \frac{n-1}{n'-1}\ell_\mu. \tag{75}$$

In this case the number of intact elements is proportional to the number of multiplications

$$\ell_{\text{id}} = \frac{n-n'}{n'-1}\ell_\mu. \tag{76}$$

3. $n' = n'_{\min} = 2 \Longrightarrow \Phi_k^{(n,2)}$ *is the $(n-1)\,\ell_\mu$-place (multiplace)* `binarizing heteromorphism`*, i.e.,*

$$k = k_{\max} = (n-1)\,\ell_\mu. \tag{77}$$

In the extreme (first and third) cases there are no restrictions on the initial arity n, while in the intermediate case n is "quantized" due to the fact that fractions in (68) and (69) should be integers. Thus, we have established a general structure and classification of heteromorphisms defined by (65). The next important issue is the preservation of special properties (associativity, commutativity, etc.), while passing from μ_n to $\mu'_{n'}$, which will further restrict the concrete shape of the main relation (65) for each choice of the `heteromorphism parameters`: arities n, n', places k, number of intacts ℓ_{id} and multiplications ℓ_μ.

ASSOCIATIVITY AND HETEROMORPHISMS

The most important property of the heteromorphism, which is needed for its next applications to representation theory, is the associativity of the final operation $\mu'_{n'}$, when the initial operation μ_n is associative. In other words, we consider here the concrete form of semigroup heteromorphisms. In general, this is a complicated task, because it is not clear from (65), which permutation in the l.h.s. should be taken to get an associative product in its r.h.s. for each set of the heteromorphism parameters. Straightforward checking of the associativity of the final operation $\mu'_{n'}$ for each permutation in the l.h.s. of (65) is almost impossible, especially for higher n. To solve this difficulty we introduce the concept of the associative polyadic quiver and special rules to construct the associative final operation $\mu'_{n'}$.

A `polyadic quiver of products` is the set of elements from G_n (presented as several copies of some matrix of the elements glued together) and arrows, such that the elements along arrows form n-ary products μ_n. For instance, for the multiplication $\mu_4 [g_1, h_2, g_2, u_1]$ the 4-adic quiver is denoted by $\{g_1 \to h_2 \to g_2 \to u_1\}$. Here the elements from G_n are arbitrary and have no connection with heteromorphisms.

Next we define `polyadic quivers` which are related to the main heteromorphism equation (65) in the following way: 1) the matrix of elements is the transposed matrix from the r.h.s. of (65), such that different letters correspond to their place in $\Phi_k^{(n,n')}$ and the low index of an element is related to its position in the $\mu'_{n'}$ product; 2) the number of polyadic quivers is ℓ_μ, which corresponds to ℓ_μ multiplications in the l.h.s. of (65); 3) the heteromorphism parameters (n, n', k, ℓ_{id} and ℓ_μ) are not arbitrary, but satisfy the arity changing formulas (68)-(69). In this way, a polyadic quiver makes for a clear visualization of the main heteromorphism equation (65); 4) The intact elements will be placed after a semicolon.

For example, the polyadic quiver $\{g_1 \to h_2 \to g_2 \to u_1; h_1, u_2\}$ corresponds to the heteromorphism with $n = 4$, $n' = 2$, $k = 3$, $\ell_{\mathrm{id}} = 2$ and $\ell_\mu = 1$ is

$$\Phi_3^{(4,2)} \begin{pmatrix} \mu_4 [g_1, h_2, g_2, u_1] \\ h_1 \\ u_2 \end{pmatrix} = \mu'_2 \left[\Phi_3^{(4,2)} \begin{pmatrix} g_1 \\ h_1 \\ u_1 \end{pmatrix}, \Phi_3^{(4,2)} \begin{pmatrix} g_2 \\ h_2 \\ u_2 \end{pmatrix} \right]. \tag{78}$$

As it is seen from (78), the product μ'_2 is not associative, if μ_4 is associative. So, not all polyadic quivers preserve associativity.

An `associative polyadic quiver` is a polyadic quiver which ensures the final associativity of $\mu'_{n'}$ in the main heteromorphism equation (65), when the initial multiplication μ_n is associative.

So, one of the associative polyadic quivers which corresponds to the same heteromorphism parameters as the non-associative quiver (78) is $\{g_1 \to h_2 \to u_1 \to g_2; h_1, u_2\}$ which corresponds to

$$\Phi_3^{(4,2)} \begin{pmatrix} \mu_4 [g_1, h_2, u_1, g_2] \\ h_1 \\ u_2 \end{pmatrix} = \mu'_2 \left[\Phi_3^{(4,2)} \begin{pmatrix} g_1 \\ h_1 \\ u_1 \end{pmatrix}, \Phi_3^{(4,2)} \begin{pmatrix} g_2 \\ h_2 \\ u_2 \end{pmatrix} \right]. \tag{79}$$

We propose a classification of associative polyadic quivers and the rules of construction of the corresponding heteromorphism equations, and use the heteromorphism parameters for the classification of ℓ_μ-ple heteromorphisms (75). In other words, we describe a consistent procedure for building the semigroup heteromorphisms.

Let us consider the first class of heteromorphisms (without intact elements $\ell_{\mathrm{id}} = 0$ or `intactless`), that is ℓ_μ-place (multiplace) homomorphisms. In the simplest case, associativity can be achieved, when all elements in a product are taken from the same row. The number of places k is not fixed by the arity relation (68) and can be arbitrary, while the arrows can have various directions. There are 2^k such combinations which preserve associativity. If the arrows have the same direction, this kind of mapping is also called a homomorphism. As an example, for $n = n' = 3$, $k = 2$, $\ell_\mu = 2$ we have

$$\Phi_2^{(3,3)} \begin{pmatrix} \mu_3 [g_1, g_2, g_3] \\ \mu_3 [h_1, h_2, h_3] \end{pmatrix} = \mu'_3 \left[\Phi_2^{(3,3)} \begin{pmatrix} g_1 \\ h_1 \end{pmatrix}, \Phi_2^{(3,3)} \begin{pmatrix} g_2 \\ h_2 \end{pmatrix}, \Phi_2^{(3,3)} \begin{pmatrix} g_3 \\ h_3 \end{pmatrix} \right]. \tag{80}$$

Note that the analogous quiver with opposite arrow directions is

$$\Phi_2^{(3,3)} \begin{pmatrix} \mu_3 [g_1, g_2, g_3] \\ \mu_3 [h_3, h_2, h_1] \end{pmatrix} = \mu'_3 \left[\Phi_2^{(3,3)} \begin{pmatrix} g_1 \\ h_1 \end{pmatrix}, \Phi_2^{(3,3)} \begin{pmatrix} g_2 \\ h_2 \end{pmatrix}, \Phi_2^{(3,3)} \begin{pmatrix} g_3 \\ h_3 \end{pmatrix} \right]. \tag{81}$$

The latter mapping was used in constructing the middle representations of ternary groups [19].

An important class of intactless heteromorphisms (with $\ell_{\mathrm{id}} = 0$) preserving associativity can be constructed using an analogy with the Post substitutions [5], and therefore we call it the `Post-like associative quiver`. The number of places k is now fixed by $k = n - 1$, while $n' = n$ and $\ell_\mu = k = n - 1$. An example of the Post-like associative quiver with the same heteromorphisms parameters as in (80)-(81) is

$$\Phi_2^{(3,3)} \begin{pmatrix} \mu_3 [g_1, h_2, g_3] \\ \mu_3 [h_1, g_2, h_3] \end{pmatrix} = \mu'_3 \left[\Phi_2^{(3,3)} \begin{pmatrix} g_1 \\ h_1 \end{pmatrix}, \Phi_2^{(3,3)} \begin{pmatrix} g_2 \\ h_2 \end{pmatrix}, \Phi_2^{(3,3)} \begin{pmatrix} g_3 \\ h_3 \end{pmatrix} \right]. \tag{82}$$

This construction appeared in the study of ternary semigroups of morphisms [101–103]. Its n-ary generalization was used in the consideration of polyadic operations on Cartesian powers [104], polyadic analogs of the Cayley and Birkhoff theorems [84, 105] and special representations of n-groups [106, 107] (where the n-group with the multiplication μ'_2 was called the `diagonal n-group`). Consider the following example.

Let Λ be the Grassmann algebra consisting of even and odd parts $\Lambda = \Lambda_{\bar 0} \oplus \Lambda_{\bar 1}$ (see e.g., [108]). The odd part can be considered as a ternary semigroup $\mathrm{G}_3^{(\bar 1)} = \langle \Lambda_{\bar 1}, \mu_3 \rangle$, its multiplication $\mu_3 : \Lambda_{\bar 1} \times \Lambda_{\bar 1} \times \Lambda_{\bar 1} \to \Lambda_{\bar 1}$ is defined by $\mu_3 [\alpha, \beta, \gamma] = \alpha \cdot \beta \cdot \gamma$, where $(\cdot)$ is multiplication in Λ and $\alpha, \beta, \gamma \in \Lambda_{\bar 1}$, so $\mathrm{G}_3^{(\bar 1)}$ is nonderived and contains no unity. The *even part can be treated as a ternary group* $\mathrm{G}_3^{(\bar{0})} = \langle \Lambda_{\bar{0}}, \mu_3' \rangle$ *with the multiplication* $\mu_3' : \Lambda_{\bar{0}} \times \Lambda_{\bar{0}} \times \Lambda_{\bar{0}} \to \Lambda_{\bar{0}}$*, defined by* $\mu_3 [a, b, c] = a \cdot b \cdot c$*, where* $a, b, c \in \Lambda_{\bar{0}}$*, thus* $\mathrm{G}_3^{(\bar{0})}$ *is derived and contains unity. We introduce the heteromorphism* $\mathrm{G}_3^{(\bar{1})} \to \mathrm{G}_3^{(\bar{0})}$ *as a mapping (2-place homomorphism)* $\Phi_2^{(3,3)} : \Lambda_{\bar{1}} \times \Lambda_{\bar{1}} \to \Lambda_{\bar{0}}$ *by the formula*

$$\Phi_2^{(3,3)} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \cdot \beta, \tag{83}$$

where $\alpha, \beta \in \Lambda_{\bar{1}}$*. It is seen that* $\Phi_2^{(3,3)}$ *defined by (83) satisfies the Post-like heteromorphism equation (82), but not the "vertical" one (80), due to the anticommutativity of odd elements from* $\Lambda_{\bar{1}}$*. In other words,* $\mathrm{G}_3^{(\bar{0})}$ *can be treated as a nontrivial example of the "diagonal" semigroup of* $\mathrm{G}_3^{(\bar{1})}$ *(according to the notation of [106, 107]).*

Note that for $k \geq 3$ there exist additional (to the above) associative quivers having the same heteromorphism parameters. For instance, when $n' = n = 4$ and $k = 3$ we have the Post-like associative quiver

$$\Phi_3^{(4,4)} \begin{pmatrix} \mu_4 [g_1, h_2, u_3, g_4] \\ \mu_4 [h_1, u_2, g_3, h_4] \\ \mu_4 [u_1, g_2, h_3, u_4] \end{pmatrix} = \mu_4' \left[\Phi_3^{(4,4)} \begin{pmatrix} g_1 \\ h_1 \\ u_1 \end{pmatrix}, \Phi_3^{(4,4)} \begin{pmatrix} g_2 \\ h_2 \\ u_2 \end{pmatrix}, \Phi_3^{(4,4)} \begin{pmatrix} g_1 \\ h_1 \\ u_1 \end{pmatrix}, \Phi_3^{(4,4)} \begin{pmatrix} g_2 \\ h_2 \\ u_2 \end{pmatrix} \right]. \tag{84}$$

Also, we have one intermediate non-Post associative quiver

$$\Phi_3^{(4,4)} \begin{pmatrix} \mu_4 [g_1, u_2, h_3, g_4] \\ \mu_4 [h_1, g_2, u_3, h_4] \\ \mu_4 [u_1, h_2, g_3, u_4] \end{pmatrix} = \mu_4' \left[\Phi_3^{(4,4)} \begin{pmatrix} g_1 \\ h_1 \\ u_1 \end{pmatrix}, \Phi_3^{(4,4)} \begin{pmatrix} g_2 \\ h_2 \\ u_2 \end{pmatrix}, \Phi_3^{(4,4)} \begin{pmatrix} g_1 \\ h_1 \\ u_1 \end{pmatrix}, \Phi_3^{(4,4)} \begin{pmatrix} g_2 \\ h_2 \\ u_2 \end{pmatrix} \right]. \tag{85}$$

The next type of heteromorphism (intermediate) is described by the equations (66)-(76), and it contains intact elements ($\ell_{\mathrm{id}} \geq 1$) and changes (decreases) arity to $n' < n$. For each fixed k the arities are not arbitrary. There are many other possibilities (using permutations and different variants of quivers) to obtain an associative final product $\mu_{n'}'$ corresponding the same heteromorphism parameters. The above examples are sufficient to understand the rules of their construction for each concrete case.

MULTIPLACE REPRESENTATIONS OF POLYADIC SYSTEMS

Representation theory (see e.g. [109]) deals with mappings from abstract algebraic systems into linear systems, such as, e.g. linear operators in vector spaces, or into general (semi)groups of transformations of some set. In our notation, this means that in the mapping of polyadic systems (50) the final multiplication $\mu_{n'}'$ is a linear map. This leads to some restrictions on the final polyadic structure $\mathrm{G}_{n'}'$, which are considered below.

Let V be a vector space over a field $\mathbb{K}$ (usually algebraically closed) and $\mathrm{End}\, V$ be a set of linear endomorphisms of V, which is in fact a binary group. In the standard way, a linear representation of a binary semigroup $\mathrm{G}_2 = \langle G; \mu_2 \rangle$ is a (1-place) map $\Pi_1 : \mathrm{G}_2 \to \mathrm{End}\, V$, such that Π_1 is a homomorphism

$$\Pi_1 (\mu_2 [g, h]) = \Pi_1 (g) * \Pi_1 (h), \tag{86}$$

where $g, h \in G$ and $(*)$ is the binary multiplication in $\mathrm{End}\, V$ (usually, it is a (semi)group with multiplication as composition of operators or product of matrices, if a basis is chosen). If G_2 is a binary group with the unity e, then we have the additional condition

$$\Pi_1 (e) = \mathrm{id}_V . \tag{87}$$

We will generalize these known formulas to the corresponding polyadic systems along with the heteromorphism concept introduced above. Our general idea is to use the heteromorphism equation (65) instead of the standard homomorphism equation (86), such that the arity of the representation will be different from the arity of the initial polyadic system $n' \neq n$.

Consider the structure of the final n'-ary multiplication $\mu_{n'}'$ in (65), taking into account that the final polyadic system $\mathrm{G}_{n'}'$ should be constructed from $\mathrm{End}\, V$. The most natural and physically applicable way is to consider the binary $\mathrm{End}\, V$ and to put $\mathrm{G}_{n'}' = \mathrm{der}_{n'} (\mathrm{End}\, V)$, as it was proposed for the ternary case in [19]. In this way $\mathrm{G}_{n'}'$ becomes a derived n'-ary (semi)group of endomorphisms of V with the multiplication $\mu_{n'}' : (\mathrm{End}\, V)^{\times n'} \to \mathrm{End}\, V$, where

$$\mu_{n'}' [v_1, \ldots, v_{n'}] = v_1 * \ldots * v_{n'}, \quad v_i \in \mathrm{End}\, V. \tag{88}$$

Because the multiplication $\mu_{n'}'$ (88) is derived and is therefore associative by definition, we may consider the associative initial polyadic systems (semigroups and groups) and the associativity preserving mappings that are the special heteromorphisms constructed in the previous section.

Let $\mathrm{G}_n = \langle G; \mu_n \rangle$ be an associative n-ary polyadic system. By analogy with (57), we introduce the following k-place mapping

$$\Pi_k^{(n,n')} : G^{\times k} \to \mathrm{End}\, V. \tag{89}$$

A multiplace representation of an associative polyadic system G_n in a vector space V is given, if there exists a k-place mapping (89) which satisfies the (associativity preserving) heteromorphism equation (65), that is

$$\Pi_k^{(n,n')}\left(\begin{array}{c}\left.\begin{array}{c}\mu_n\left[g_1,\ldots,g_n\right],\\ \vdots\\ \mu_n\left[g_{n(\ell_\mu-1)},\ldots,g_{n\ell_\mu}\right]\end{array}\right\}\ell_\mu\\ \left.\begin{array}{c}g_{n\ell_\mu+1},\\ \vdots\\ g_{n\ell_\mu+\ell_{\mathrm{id}}}\end{array}\right\}\ell_{\mathrm{id}}\end{array}\right)=\overbrace{\Pi_k^{(n,n')}\left(\begin{array}{c}g_1\\ \vdots\\ g_k\end{array}\right)*\ldots*\Pi_k^{(n,n')}\left(\begin{array}{c}g_{k(n'-1)}\\ \vdots\\ g_{kn'}\end{array}\right)}^{n'},\tag{90}$$

where $\mu_n^{(\ell_\mu,\ell_{\mathrm{id}})}$ is given by (63), ℓ_μ and ℓ_{id} are the numbers of multiplications and intact elements in the l.h.s. of (90), respectively.

The exact permutation in the l.h.s. of (90) is given by the associative quiver presented in the previous section. The representation parameters (n, n', k, ℓ_μ and ℓ_{id}) in (90) are the same as the heteromorphism parameters, and they satisfy the same arity changing formulas (68) and (69). Therefore, a general classification of multiplace representations can be done by analogy with that of the heteromorphisms (74)–(77) as follows:

1. The hom-like multiplace representation which is a multiplace homomorphism with $n'=n'_{\max}=n$, without intact elements $l_{\mathrm{id}}=l_{\mathrm{id}}^{(\min)}=0$, and minimal number of places
$$k=k_{\min}=\ell_\mu.\tag{91}$$
2. The intact element multiplace representation which is the intermediate heteromorphism with $2<n'<n$ and the number of intact elements is
$$l_{\mathrm{id}}=\frac{n-n'}{n'-1}\ell_\mu.\tag{92}$$
3. The binary multiplace representation which is a binarizing heteromorphism (77) with $n'=n'_{\min}=2$, the maximal number of intact elements $l_{\mathrm{id}}^{(\max)}=(n-2)\,\ell_\mu$ and maximal number of places
$$k=k_{\max}=(n-1)\,\ell_\mu.\tag{93}$$

The multiplace representations for n-ary semigroups have no additional defining relations, as compared with (90). In case of n-ary groups, we need an analog of the “normalizing” relation (87). If the n-ary group has the unity e, then one can put

$$\Pi_k^{(n,n')}\left(\left.\begin{array}{c}e\\ \vdots\\ e\end{array}\right\}k\right)=\mathrm{id}_V\,.\tag{94}$$

If there is no unity at all, one can “normalize” the multiplace representation, using analogy with (87) in the form

$$\Pi_1\left(h^{-1}*h\right)=\mathrm{id}_V,\tag{95}$$

as follows

$$\Pi_k^{(n,n')}\left(\begin{array}{c}\left.\begin{array}{c}\bar{h}\\ \vdots\\ \bar{h}\end{array}\right\}\ell_\mu\\ \left.\begin{array}{c}h\\ \vdots\\ h\end{array}\right\}\ell_{\mathrm{id}}\end{array}\right)=\mathrm{id}_V,\tag{96}$$

for all $h\in\mathrm{G}_n$, where $\bar{h}$ is the querelement of h. The latter ones can be placed on any places in the l.h.s. of (96) due to the Dörnte identities. Also, the multiplications in the l.h.s. of (90) can change their place due to the same reason.

A general form of multiplace representations can be found by applying the Dörnte identities to each n-ary product in the l.h.s. of (90). Then, using (96) we have schematically

$$\Pi_k^{(n,n')}\left(\begin{array}{c}g_1\\ \vdots\\ g_k\end{array}\right)=\Pi_k^{(n,n')}\left(\begin{array}{c}t_1\\ \vdots\\ t_{\ell_\mu}\\ \left.\begin{array}{c}g\\ \vdots\\ g\end{array}\right\}\ell_{\mathrm{id}}\end{array}\right),\tag{97}$$

where g is an arbitrary fixed element of the n-ary group and

$$t_a = \mu_n [g_{a1}, \dots, g_{an-1}, \bar{g}], \quad a = 1, \dots, \ell_\mu. \tag{98}$$

This is the special shape of some multiplace representations, while the concrete formulas should be obtained in each case separately. Nevertheless, some conclusions can be drawn from (97). Firstly, the equivalence classes on which $\Pi_k^{(n,n')}$ is constant are determined by fixing $\ell_\mu + 1$ elements, i.e. by the surface $t_a = const$, $g = const$. Secondly, some k-place representations of a n-ary group can be reduced to ℓ_μ-place representations of its retract. In the case $\ell_\mu = 1$, multiplace representations of a n-ary group derived from a binary group correspond to ordinary representations of the latter (see [19, 53]).

The above formulas describe various properties of multiplace representations, but they give no idea of how to build representations for concrete polyadic systems. The most common method of representation construction uses the concept of a group action on a set. Below we extend this concept to the multiplace case, as it was done above for homomorphisms and representations.

MULTIACTIONS AND G-SPACES

Let $\mathrm{G}_n = \langle G; \mu_n \rangle$ be a polyadic system and X be a set. A (left) 1-place action of G_n on X is the external binary operation $\boldsymbol{\rho}_1^{(n)} : G \times \mathsf{X} \to \mathsf{X}$ such that it is consistent with the multiplication μ_n, i.e. composition of the binary operations $\boldsymbol{\rho}_1 \{g|\mathsf{x}\}$ gives the n-ary product, that is,

$$\boldsymbol{\rho}_1^{(n)} \{\mu_n [g_1, \dots g_n] |\mathsf{x}\} = \boldsymbol{\rho}_1^{(n)} \left\{g_1 | \boldsymbol{\rho}_1^{(n)} \left\{g_2 | \dots | \boldsymbol{\rho}_1^{(n)} \{g_n | \mathsf{x}\}\right\} \dots\right\}, \quad g_1, \dots, g_n \in G, \ \mathsf{x} \in \mathsf{X}. \tag{99}$$

If the polyadic system is a n-ary group, then in addition to (99) it is implied the there exist such $e_x \in G$ (which may or may not coincide with the unity of G_n) that $\boldsymbol{\rho}_1^{(n)} \{e_x|\mathsf{x}\} = \mathsf{x}$ for all $\mathsf{x} \in \mathsf{X}$, and the mapping $\mathsf{x} \mapsto \boldsymbol{\rho}_1^{(n)} \{e_x|\mathsf{x}\}$ is a bijection of X. The right 1-place actions of G_n on X are defined in a symmetric way, and therefore we will consider below only one of them. Obviously, we cannot compose $\boldsymbol{\rho}_1^{(n)}$ and $\boldsymbol{\rho}_1^{(n')}$ with $n \neq n'$. Usually X is called a G-set or G-space depending on its properties (see, e.g., [110]).

The application of the 1-place action defined by (99) to the representation theory of n-ary groups gave mostly repetitions of the ordinary (binary) group representation results (except for trivial b-derived ternary groups) [20]. Also, it is obviously seen that the construction (99) with the binary external operation $\boldsymbol{\rho}_1$ cannot be applied for studying the most important regular representations of polyadic systems, when the X coincides with G_n itself and the action arises from translations.

Here we introduce the multiplace concept of action for polyadic systems, which is consistent with heteromorphisms and multiplace representations. Then we will show how it naturally appears when $\mathsf{X} = \mathrm{G}_n$ and apply it to construct examples of representations including the regular ones.

For a polyadic system $\mathrm{G}_n = \langle G; \mu_n \rangle$ and a set X we introduce an external polyadic operation

$$\boldsymbol{\rho}_k : G^{\times k} \times \mathsf{X} \to \mathsf{X}, \tag{100}$$

which is called a (left) k-`place action` or `multiaction`. To generalize the 1-action composition (99), we use the analogy with multiplication laws of the heteromorphisms (65) and the multiplace representations (90) and propose (schematically)

$$\boldsymbol{\rho}_k^{(n)} \left\{ \begin{array}{c} \left.\begin{array}{c} \mu_n [g_1, \dots, g_n], \\ \vdots \\ \mu_n \left[g_{n(\ell_\mu - 1)}, \dots, g_{n\ell_\mu}\right] \end{array}\right\} \ell_\mu \\ \left.\begin{array}{c} g_{n\ell_\mu + 1}, \\ \vdots \\ g_{n\ell_\mu + \ell_{\mathrm{id}}} \end{array}\right\} \ell_{\mathrm{id}} \end{array} \middle| \, \mathsf{x} \right\} = \boldsymbol{\rho}_k^{(n)} \overbrace{\left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \dots \middle| \rho_k^{(n)} \left\{ \begin{array}{c} g_{k(n'-1)} \\ \vdots \\ g_{kn'} \end{array} \middle| \, \mathsf{x} \right\} \dots \right\}}^{n'}. \tag{101}$$

The connection between all the parameters here is the same as in the arity changing formulas (68)–(69). Composition of mappings is associative, and therefore in concrete cases we can use the associative quiver technique, as it is described in the previous sections. If G_n is n-ary group, then we should add to (101) the "normalizing" relations analogous with (94) or (96). So, if there is a unity $e \in \mathrm{G}_n$, then

$$\boldsymbol{\rho}_k^{(n)} \left\{ \begin{array}{c} e \\ \vdots \\ e \end{array} \middle| \, \mathsf{x} \right\} = \mathsf{x}, \quad \text{for all } \mathsf{x} \in \mathsf{X}. \tag{102}$$

In terms of the querelement, the normalization has the form

$$\boldsymbol{\rho}_k^{(n)} \left\{ \begin{array}{c} \left.\begin{array}{c} \bar{h} \\ \vdots \\ \bar{h} \end{array}\right\} \ell_\mu \\ \left.\begin{array}{c} h \\ \vdots \\ h \end{array}\right\} \ell_{\mathrm{id}} \end{array} \middle| \, \mathsf{x} \right\} = \mathsf{x}, \quad \text{for all } \mathsf{x} \in \mathsf{X} \text{ and for all } h \in \mathrm{G}_n. \tag{103}$$

The multiaction $\rho_k^{(n)}$ is `transitive`, if any two points x and y in X can be "connected" by $\rho_k^{(n)}$, i.e. there exist $g_1, \ldots, g_k \in G_n$ such that

$$\rho_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \, \mathsf{x} \right\} = \mathsf{y}. \tag{104}$$

If $g_1, \ldots, g_k$ are unique, then $\rho_k^{(n)}$ is `sharply transitive`. The subset of X, in which any points are connected by (104) with fixed $g_1, \ldots, g_k$ can be called the `multiorbit` of X. If there is only one multiorbit, then we call X the `heterogenous` G-space (by analogy with the homogeneous one). By analogy with the (ordinary) 1-place actions, we define a G-equivariant map Ψ between two G-sets X and Y by (in our notation)

$$\Psi \left(\rho_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \, \mathsf{x} \right\} \right) = \boldsymbol{\rho}_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \, \Psi(\mathsf{x}) \right\} \in \mathsf{Y}, \tag{105}$$

which makes G-space into a category (for details, see, e.g., [110]). In the particular case, when X is a vector space over $\mathbb{K}$, the multiaction (100) can be called a `multi`-G-module which satisfies (102) and the additional (linearity) conditions

$$\rho_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \, a\mathsf{x} + b\mathsf{y} \right\} = a\boldsymbol{\rho}_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \, \mathsf{x} \right\} + b\boldsymbol{\rho}_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \, \mathsf{y} \right\}, \tag{106}$$

where $a, b \in \mathbb{K}$. Then, comparing (90) and (101) we can define a multiplace representation as a multi-G-module by the following formula

$$\Pi_k^{(n,n')} \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix} (\mathsf{x}) = \boldsymbol{\rho}_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \, \mathsf{x} \right\}. \tag{107}$$

In a similar way, one can generalize to polyadic systems many other notions from group action theory [109].

REGULAR MULTIACTIONS

The most important role in the study of polyadic systems is played by the case, when X $=\mathrm{G}_n$, and the multiaction coincides with the n-ary analog of translations [111], so called i-translations [56]. In the binary case, ordinary translations lead to regular representations [109], and therefore we call such an action a `regular multiaction` $\boldsymbol{\rho}_k^{reg(n)}$. In this connection, the analog of the Cayley theorem for n-ary groups was obtained in [105,112]. Now we will show in examples, how the regular multiactions can arise from i-translations.

Let G_3 *be a ternary semigroup,* $k = 2$*, and* X $=\mathrm{G}_3$*, then 2-place (left) action can be defined as*

$$\boldsymbol{\rho}_2^{reg(3)} \left\{ \begin{array}{c} g \\ h \end{array} \middle| \, u \right\} \stackrel{def}{=} \mu_3 [g, h, u]. \tag{108}$$

This gives the following composition law for two regular multiactions

$$\boldsymbol{\rho}_2^{reg(3)} \left\{ \begin{array}{c} g_1 \\ h_1 \end{array} \middle| \, \boldsymbol{\rho}_2^{reg(3)} \left\{ \begin{array}{c} g_2 \\ h_2 \end{array} \middle| \, u \right\} \right\} = \mu_3 [g_1, h_1, \mu_3 [g_2, h_2, u]]$$
$$= \mu_3 [\mu_3 [g_1, h_1, g_2], h_2, u] = \boldsymbol{\rho}_2^{reg(3)} \left\{ \begin{array}{c} \mu_3 [g_1, h_1, g_2] \\ h_2 \end{array} \middle| \, u \right\}. \tag{109}$$

Thus, using the regular 2-action (108) *we have, in fact, derived the associative quiver corresponding to (59).*

The formula (108) can be simultaneously treated as a 2-translation [56]. In this way, the following `left regular multiaction`

$$\boldsymbol{\rho}_k^{reg(n)}\left\{\begin{array}{c|c} g_1 & \\ \vdots & h \\ g_k & \end{array}\right\} \overset{def}{=} \mu_n\left[g_1,\ldots,g_k,h\right], \tag{110}$$

corresponds to (62), where in the r.h.s. there is the i-translation with $i=n$. The `right regular multiaction` corresponds to the i-translation with $i=1$. The binary composition of the left regular multiactions corresponds to (62). In general, the value of i fixes the minimal final arity n'_{reg}, which differs for even and odd values of the initial arity n.

It follows from (110) that for regular multiactions the number of places is fixed

$$k_{reg}=n-1, \tag{111}$$

and the arity changing formulas (68)–(69) become

$$n'_{reg}=n-\ell_{\mathrm{id}} \tag{112}$$

$$n'_{reg}=\ell_\mu+1. \tag{113}$$

From (112)–(113) we conclude that for any n a regular multiaction having one multiplication $\ell_\mu=1$ is binarizing and has $n-2$ intact elements. For $n=3$ see (109). Also, it follows from (112) that for regular multiactions the number of intact elements gives exactly the difference between initial and final arities.

If the initial arity is odd, then there exists a special `middle regular multiaction` generated by the i-translation with $i=(n+1)\,/\,2$. For $n=3$ the corresponding associative quiver is (81) and such 2-actions were used in [19] to construct middle representations of ternary groups, which did not change arity ($n'=n$). Here we give a more complicated example of a middle regular multiaction, which can contain intact elements and can therefore change arity.

Let us consider 5-ary semigroup and the following middle 4-action

$$\boldsymbol{\rho}_4^{reg(5)}\left\{\begin{array}{c|c} g & \\ h & \\ u & s \\ v & \end{array}\right\}=\mu_5\left[g,h,\overset{i=3}{\overset{\downarrow}{s}},u,v\right]. \tag{114}$$

Using (113) we observe that there are two possibilities for the number of multiplications $\ell_\mu=2,4$. The last case $\ell_\mu=4$ is similar to the vertical associative quiver (81), but with a more complicated l.h.s., that is

$$\boldsymbol{\rho}_4^{reg(5)}\left\{\begin{array}{c|c} \mu_5\left[g_1,h_1,g_2,h_2,g_3\right] & \\ \mu_5\left[h_3,g_4,h_4,g_5,h_5\right] & s \\ \mu_5\left[u_5,v_5,u_4,v_4,u_3\right] & \\ \mu_5\left[v_3,u_2,v_2,u_1,v_1\right] & \end{array}\right\}=$$
$$\boldsymbol{\rho}_4^{reg(5)}\left\{\begin{array}{c|c} g_1 & \\ h_1 & \boldsymbol{\rho}_4^{reg(5)}\left\{\begin{array}{c|c} g_2 & \\ h_2 & \boldsymbol{\rho}_4^{reg(5)}\left\{\begin{array}{c|c} g_3 & \\ h_3 & \boldsymbol{\rho}_4^{reg(5)}\left\{\begin{array}{c|c} g_4 & \\ h_4 & \boldsymbol{\rho}_4^{reg(5)}\left\{\begin{array}{c|c} g_5 & \\ h_5 & s \\ u_5 & \\ v_5 & \end{array}\right\} \\ u_4 & \\ v_4 & \end{array}\right\} \\ u_3 & \\ v_3 & \end{array}\right\} \\ u_2 & \\ v_2 & \end{array}\right\} \\ u_1 & \\ v_1 & \end{array}\right\}. \tag{115}$$

Now we have an additional case with two intact elements ℓ_{id} and two multiplications $\ell_\mu=2$ as

$$\boldsymbol{\rho}_4^{reg(5)}\left\{\begin{array}{c|c} \mu_5\left[g_1,h_1,g_2,h_2,g_3\right] & \\ h_3 & s \\ \mu_5\left[h_3,v_3,u_2,v_2,u_1\right] & \\ v_1 & \end{array}\right\}=\boldsymbol{\rho}_4^{reg(5)}\left\{\begin{array}{c|c} g_1 & \\ h_1 & \boldsymbol{\rho}_4^{reg(5)}\left\{\begin{array}{c|c} g_2 & \\ h_2 & \boldsymbol{\rho}_4^{reg(5)}\left\{\begin{array}{c|c} g_3 & \\ h_3 & s \\ u_3 & \\ v_3 & \end{array}\right\} \\ u_2 & \\ v_2 & \end{array}\right\} \\ u_1 & \\ v_1 & \end{array}\right\}, \tag{116}$$

with arity changing from $n=5$ to $n'_{reg}=3$. In addition to (116) we have 3 more possible regular multiactions due to the associativity of μ_5, when the multiplication brackets in the sequences of 6 elements in the first two rows and the second two ones can be shifted independently.

For $n>3$, in addition to left, right and middle multiactions, there exist intermediate cases. First, observe that the i-translations with $i=2$ and $i=n-1$ immediately fix the final arity $n'_{reg}=n$. Therefore, the composition of multiactions will be similar to (115), but with some permutations in the l.h.s.

Now we consider some multiplace analogs of regular representations of binary groups [109]. The straightforward generalization is to consider the previously introduced regular multiactions (110) in the r.h.s. of (107). Let G_n be a finite polyadic associative system and $\mathbb{K}\mathrm{G}_n$ be a vector space spanned by G_n (some properties of n-ary group rings were considered in [113, 114]). This means that any element of $\mathbb{K}\mathrm{G}_n$ can be uniquely presented in the form $w = \sum_l a_l \cdot h_l$, $a_l \in \mathbb{K}$, $h_l \in G$. Then, using (110) and (107) we define the i-regular k-place representation by

$$\Pi_k^{reg(i)} \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix} (w) = \sum_l a_l \cdot \mu_{k+1} [g_1 \ldots g_{i-1} h_l g_{i+1} \ldots g_k] . \tag{117}$$

Comparing (110) and (117) one can conclude that all the general properties of multiplace regular representations are similar to those of the regular multiactions. If $i = 1$ or $i = k$, the multiplace representation is called a right or left regular representation respectively. If k is even, the representation with $i = k/2 + 1$ is called a middle regular representation. The case $k = 2$ was considered in [19] for ternary groups.

MULTIPLACE REPRESENTATIONS OF TERNARY GROUPS

Let us consider the case $n = 3$, $k = 2$ in more detail, paying attention to its special peculiarities, which corresponds to the 2-place (bi-element) representations of ternary groups [19]. Let V be a vector space over $\mathbb{K}$ and $\mathrm{End}\, V$ be a set of linear endomorphisms of V. From now on we denote the ternary multiplication by square brackets only, as follows $\mu_3 [g_1, g_2, g_3] \equiv [g_1 g_2 g_3]$, and use the "horizontal" notation $\Pi \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \equiv \Pi (g_1, g_2)$.

A left representation of a ternary group $G, [\,])$ in V is a map $\Pi^L : G \times G \to \mathrm{End}\, V$ such that

$$\Pi^L (g_1, g_2) \circ \Pi^L (g_3, g_4) = \Pi^L ([g_1 g_2 g_3] , g_4) , \tag{118}$$

$$\Pi^L (g, \overline{g}) = \mathrm{id}_V , \tag{119}$$

where $g, g_1, g_2, g_3, g_4 \in G$.

Replacing in (119) g by $\overline{g}$ we obtain $\Pi^L (\overline{g}, g) = id_V$, which means that in fact (119) has the form $\Pi^L (\overline{g}, g) = \Pi^L (g, \overline{g}) = \mathrm{id}_V, \ \ \forall g \in G$. Note that the axioms considered in the above definition are the natural ones satisfied by left multiplications $g \mapsto [abg]$. For all $g_1, g_2, g_3, g_4 \in G$ we have

$$\Pi^L ([g_1 g_2 g_3] , g_4) = \Pi^L (g_1, [g_2 g_3 g_4]) .$$

For all $g, h, u \in G$ we have

$$\Pi^L (g, h) = \Pi^L ([g u \overline{u}], h) = \Pi^L (g, u) \circ \Pi^L (\overline{u}, h) \tag{120}$$

and

$$\Pi^L (g, u) \circ \Pi^L (\overline{u}, \overline{g}) = \Pi^L (\overline{u}, \overline{g}) \circ \Pi^L (g, u) = \mathrm{id}_V , \tag{121}$$

and therefore every $\Pi^L (g, u)$ is invertible and $\left(\Pi^L (g, u)\right)^{-1} = \Pi^L (\overline{u}, \overline{g})$. This means that any left representation gives a representation of a ternary group by a binary group [19]. If the ternary group is medial, then

$$\Pi^L (g_1, g_2) \circ \Pi^L (g_3, g_4) = \Pi^L (g_3, g_4) \circ \Pi^L (g_1, g_2) ,$$

i.e. the group so obtained is commutative. If the ternary group $\langle G, [\,] \rangle$ is commutative, then also $\Pi^L (g, h) = \Pi^L (h, g)$, because

$$\Pi^L (g, h) = \Pi^L (g, h) \circ \Pi^L (g, \overline{g}) = \Pi^L ([g\, h\, g] , \overline{g}) = \Pi^L ([h\, g\, g] , \overline{g}) = \Pi^L (h, g) \circ \Pi^L (g, \overline{g}) = \Pi^L (h, g) .$$

In the case of a commutative and idempotent ternary group any of its left representations is idempotent and $\left(\Pi^L (g, h)\right)^{-1} = \Pi^L (g, h)$, so that commutative and idempotent ternary groups are represented by Boolean groups.

Let Let $\langle G, [\,] \rangle = \mathrm{der}\, (G, \odot)$ be a ternary group derived from a binary group $\langle G, \odot \rangle$, then there is one-to-one correspondence between representations of $(G, \odot)$ and left representations of $(G, [\,])$.

Indeed, because $(G, [\,]) = \mathrm{der}\, (G, \odot)$, then $g \odot h = [geh]$ and $\overline{e} = e$, where e is unity of the binary group $(G, \odot)$. If $\pi \in \mathrm{Rep}\, (G, \odot)$, then (as it is not difficult to see) $\Pi^L (g, h) = \pi (g) \circ \pi (h)$ is a left representation of $\langle G, [\,] \rangle$. Conversely, if Π^L is a left representation of $\langle G, [\,] \rangle$ then $\pi (g) = \Pi^L (g, e)$ is a representation of $(G, \odot)$. Moreover, in this case $\Pi^L (g, h) = \pi (g) \circ \pi (h)$, because we have

$$\Pi^L (g, h) = \Pi^L (g, [ehe]) = \Pi^L ([geh], e) = \Pi^L (g, e) \circ \Pi^L (h, e) = \pi (g) \circ \pi (h) .$$

Let $(G, [\,])$ be a ternary group and $(G \times G, *)$ be a semigroup used in the construction of left representations. According to Post [5] one says that two pairs (a, b), (c, d) of elements of G are equivalent, if there exists an element $g \in G$ such that

$[abg] = [cdg]$. Using a covering group we can see that if this equation holds for some $g \in G$, then it holds also for all $g \in G$. This means that

$$\Pi^L(a,b) = \Pi^L(c,d) \Longleftrightarrow (a,b) \sim (c,d),$$

i.e.

$$\Pi^L(a,b) = \Pi^L(c,d) \Longleftrightarrow [abg] = [cdg]$$

for some $g \in G$. Indeed, if $[abg] = [cdg]$ holds for some $g \in G$, then

$$\Pi^L(a,b) = \Pi^L(a,b) \circ \Pi^L(g,\overline{g}) = \Pi^L([abg],\, \overline{g})$$
$$= \Pi^L([cdg],\, \overline{g}) = \Pi^L(c,d) \circ \Pi^L(g,\overline{g}) = \Pi^L(c,d).$$

By analogy we can define

A `right representation` of a ternary group $(G, [\,])$ in V is a map $\Pi^R : G \times G \to \operatorname{End} V$ such that

$$\Pi^R(g_3,g_4) \circ \Pi^R(g_1,g_2) = \Pi^R(g_1,[g_2g_3g_4]), \quad (122)$$
$$\Pi^R(g,\overline{g}) = \mathrm{id}_V, \quad (123)$$

where $g, g_1, g_2, g_3, g_4 \in G$.

From (122)-(123) it follows that

$$\Pi^R(g,h) = \Pi^R(g,[u\,\overline{u}\,h]) = \Pi^R(\overline{u},h) \circ \Pi^R(g,u). \quad (124)$$

It is easy to check that $\Pi^R(g,h) = \Pi^L(\overline{h},\overline{g}) = \left(\Pi^L(g,h)\right)^{-1}$. So it is sufficient to consider only left representations (as in the binary case). Consider the following example of a group algebra ternary generalization [19].

Let G be a ternary group and $\mathbb{K}G$ be a vector space spanned by G, which means that any element of $\mathbb{K}G$ can be uniquely presented in the form $t = \sum_{i=1}^{n} k_i h_i$, $k_i \in \mathbb{K}$, $h_i \in G$, $n \in \mathbb{N}$ (we do not assume that G has finite rank). Then left and right regular representations are defined by

$$\Pi^L_{reg}(g_1,g_2)\, t = \sum_{i=1}^{n} k_i\,[g_1g_2h_i], \quad (125)$$
$$\Pi^R_{reg}(g_1,g_2)\, t = \sum_{i=1}^{n} k_i\,[h_ig_1g_2]. \quad (126)$$

Let us construct the middle representations as follows.

A `middle representation` of a ternary group $\langle G, [\,]\rangle$ in V is a map $\Pi^M : G \times G \to \operatorname{End} V$ such that

$$\Pi^M(g_3,h_3) \circ \Pi^M(g_2,h_2) \circ \Pi^M(g_1,h_1) = \Pi^M([g_3g_2g_1],[h_1h_2h_3]), \quad (127)$$
$$\Pi^M(g,h) \circ \Pi^M(\overline{g},\overline{h}) = \Pi^M(\overline{g},\overline{h}) \circ \Pi^M(g,h) = \mathrm{id}_V \quad (128)$$

It can be seen that a middle representation is a ternary group homomorphism $\Pi^M : G \times G^{op} \to \operatorname{der}\operatorname{End} V$. Note that instead of (128) one can use $\Pi^M(g,\overline{h}) \circ \Pi^M(\overline{g},h) = id_V$ after changing g to $\overline{g}$ and taking into account that $g = \overline{\overline{g}}$. In the case of idempotent elements g and h we have $\Pi^M(g,h) \circ \Pi^M(g,h) = id_V$, which means that the matrices Π^M are Boolean. Thus all middle representation matrices of idempotent ternary groups are Boolean. The composition $\Pi^M(g_1,h_1) \circ \Pi^M(g_2,h_2)$ is not a middle representation, but the following proposition nevertheless holds.

Let Π^M be a middle representation of a ternary group $\langle G, [\,]\rangle$, then, if $\Pi^L_u(g,h) = \Pi^M(g,u) \circ \Pi^M(h,\overline{u})$ is a left representation of $\langle G, [\,]\rangle$, then $\Pi^L_u(g,h) \circ \Pi^L_{u'}(g',h') = \Pi^L_{u'}([ghu'],h')$, and, if $\Pi^R_u(g,h) = \Pi^M(u,h) \circ \Pi^M(\overline{u},g)$ is a right representation of $\langle G, [\,]\rangle$, then $\Pi^R_u(g,h) \circ \Pi^R_{u'}(g',h') = \Pi^R_u(g,[hg'h'])$. In particular, Π^L_u (Π^R_u) is a family of left (right) representations.

If a middle representation Π^M of a ternary group $\langle G, [\,]\rangle$ satisfies $\Pi^M(g,\overline{g}) = \mathrm{id}_V$ for all $g \in G$, then it is a left and a right representation and $\Pi^M(g,h) = \Pi^M(h,g)$ for all $g,h \in G$. Note that in general $\Pi^M_{reg}(g,\overline{g}) \neq \mathrm{id}$. For regular representations we have the following commutation relations

$$\Pi^L_{reg}(g_1,h_1) \circ \Pi^R_{reg}(g_2,h_2) = \Pi^R_{reg}(g_2,h_2) \circ \Pi^L_{reg}(g_1,h_1).$$

Let $\langle G, [\,]\rangle$ be a ternary group and let $\left\langle G \times G, [\,]'\right\rangle$ be a ternary group used in the construction of the middle representation. In $\langle G, [\,]\rangle$, and consequently in $\left\langle G \times G, [\,]'\right\rangle$, we define the relation

$$(a,b) \sim (c,d) \Longleftrightarrow [aub] = [cud]$$

for all $u \in G$. It is not difficult to see that this relation is a congruence in the ternary group $\left\langle G\times G,[\,]'\right\rangle$. For regular representations $\Pi_{reg}^{M}(a,b)=\Pi_{reg}^{M}(c,d)$ if $(a,b)\sim(c,d)$. We have the following relation

$$a\approx a'\Longleftrightarrow a=[\overline{g}a'g]\ \text{ for some }\ g\in G$$

or equivalently

$$a\approx a'\Longleftrightarrow a'=[ga\overline{g}]\ \text{ for some }\ g\in G.$$

It is not difficult to see that it is an equivalence relation on $\langle G,[\,]\rangle$, moreover, if $\langle G,[\,]\rangle$ is medial, then this relation is a congruence.

Let $\left\langle G\times G,[\,]'\right\rangle$ be a ternary group used in a construction of middle representations, then

$$\begin{aligned}(a,b)\approx(a',b)\Longleftrightarrow a'=[ga\overline{g}]\ \text{ and}\\ b'=[hb\overline{b}\,]\ \text{ for some }\ (g,h)\in G\times G\end{aligned}$$

is an equivalence relation on $\left\langle G\times G,[\,]'\right\rangle$. Moreover, if $(G,[\,])$ is medial, then this relation is a congruence. Unfortunately, however it is a weak relation. In a ternary group $\mathbb{Z}_3$, where $[ghu]=(g-h+u)\,(\mathrm{mod}\,3)$ we have only one class, i.e. all elements are equivalent. In $\mathbb{Z}_4$ with the operation $[ghu]=(g+h+u+1)\,(\mathrm{mod}\ 4)$ we have $a\approx a'\Longleftrightarrow a=a'$. However, for this relation the following statement holds. If $(a,b)\approx(a',b')$, then

$$\mathrm{tr}\,\Pi^{M}(a,b)=\mathrm{tr}\,\Pi^{M}(a',b').$$

We have $\mathrm{tr}(AB)=\mathrm{tr}(BA)$ for all $A,B\in\mathrm{End}V$, and

$$\begin{aligned}\mathrm{tr}\,\Pi^{M}(a,b)&=\mathrm{tr}\,\Pi^{M}([ga'\overline{g}],[hb'\overline{h}]\,)=\mathrm{tr}\left(\Pi^{M}(g,\overline{h})\circ\Pi^{M}(a',b')\circ\Pi^{M}(\overline{g},h)\right)\\&=\mathrm{tr}\left(\Pi^{M}(g,\overline{h})\circ\Pi^{M}(\overline{g},h)\circ\Pi^{M}(a',b')\right)=\mathrm{tr}\left(id_V\circ\Pi^{M}(a'b')\right)=\mathrm{tr}\,\Pi^{M}(a',b')\end{aligned}$$

In our derived case the connection with standard group representations is given by the following. Let $(G,\odot)$ be a binary group, and the ternary derived group as $\langle G,[\,]\rangle=\mathrm{der}\ (G,\odot)$. There is one-to-one correspondence between a pair of commuting binary group representations and a middle ternary derived group representation. Indeed, let $\pi,\rho\in Rep\,(G,\odot)$, $\pi\,(g)\circ\rho\,(h)=\rho\,(h)\circ\pi\,(g)$ and $\Pi^{L}\in Rep\,(G,[\,])$. We take

$$\Pi^{M}\,(g,h)=\pi\,(g)\circ\rho\left(h^{-1}\right),\ \ \pi\,(g)=\Pi^{M}\,(g,e)\,,\ \ \rho\,(g)=\Pi^{M}\,(e,\overline{g})\,.$$

Then using (127) we prove the needed representation laws.

Let $\langle G,[\,]\rangle$ be a fixed ternary group, $\left\langle G\times G,[\,]'\right\rangle$ a corresponding ternary group used in the construction of middle representations, $\left((G\times G)^{*},\circledast\right)$ a covering group of $\left\langle G\times G,[\,]'\right\rangle$, $(G\times G,\diamond)=\mathrm{ret}_{(a,b)}(G\times G,\langle\,\rangle)$. If $\Pi^{M}(a,b)$ is a middle representation of $\langle G,[\,]\rangle$, then π defined by

$$\pi(g,h,0)=\Pi^{M}(g,h),\quad \pi(g,h,1)=\Pi^{M}(g,h)\circ\Pi^{M}(a,b)$$

is a representation of the covering group [5]. Moreover

$$\rho(g,h)=\Pi^{M}(g,h)\circ\Pi^{M}(a,b)=\pi(g,h,1)$$

is a representation of the above retract induced by (a,b). Indeed, $(\overline{a},\overline{b})$ is the identity of this retract and $\rho(\overline{a},\overline{b})=\Pi^{M}(\overline{a},\overline{b})\circ\Pi^{M}(a,b)=\mathrm{id}_V$. Similarly

$$\begin{aligned}\rho\left((g,h)\diamond(u,u)\right)&=\rho\left(\langle(g,h),(a,b),(u,u)\rangle\right)=\rho\left([gau],[ubh]\right)=\Pi^{M}\left([gau],[ubh]\right)\right)\circ\Pi^{M}(a,b)\\&=\Pi^{M}(g,h)\circ\Pi^{M}(a,b)\circ\Pi^{M}(u,u)\circ\Pi^{M}(a,b)=\rho(g,h)\circ\rho(u,u)\end{aligned}$$

But $\tau(g)=(g,\overline{g})$ is an embedding of $(G,[\ ])$ into $\left\langle G\times G,[\,]'\right\rangle$. Hence μ defined by $\mu(g,0)=\Pi^{M}(g,\overline{g})$ and $\mu(g,1)=\Pi^{M}(g,\overline{g})\circ\Pi^{M}(a,\overline{a})$ is a representation of a covering group G^{*} for $(G,[\ ])$ (see the Post theorem [5] for $a=c$). On the other hand, $\beta(g)=\Pi^{M}(g,\overline{g})\circ\Pi^{M}(a,\overline{a})$ is a representation of a binary retract $(G,\cdot\,)=\mathrm{ret}_a(G,[\,])$. Thus β can induce some middle representation of $(G,[\,])$ (by the Gluskin-Hosszú theorem [7]).

Note that in the ternary group of quaternions $\langle\mathbb{K},[\,]\rangle$ (with norm 1), where $[ghu]=ghu(-1)=-ghu$ and gh is the multiplication of quaternions (-1 is a central element) we have $\overline{1}=-1$, $\overline{-1}=1$ and $\overline{g}=g$ for others. In $\left\langle\mathbb{K}\times\mathbb{K},[\,]'\right\rangle$ we have $(a,b)\sim(-a,-b)$ and $(a,-b)\sim(-a,b)$, which gives 32 two-element equivalence classes. The embedding $\tau(g)=(g,\overline{g})$ suggest that $\Pi^{M}(i,i)=\pi(i)\neq\pi(-i)=\Pi^{M}(-i,-i)$. Generally $\Pi^{M}(a,b)\neq\Pi^{M}(-a,-b)$ and $\Pi^{M}(a,-b)\neq\Pi^{M}(-a,b)$.

The relation $(a,b) \sim (c,d) \Longleftrightarrow [abg] = [cdg]$ for all $g \in G$ is a congruence on $(G \times G, *)$. Note that this relation can be defined as "for some g". Indeed, using a covering group we can see that if $[abg] = [cdg]$ holds for some g then it holds also for all g. Thus $\pi^L(a,b) = \Pi^L(c,d) \Longleftrightarrow (a,b) \sim (c,d)$. Indeed

$$\Pi^L(a,b) = \Pi^L(a,b) \circ \Pi^L(g,\overline{g}) = \Pi^L([a\, b\, g], \overline{g})$$
$$= \Pi^L([c\, d\, g], \overline{g}) = \Pi^L(c,d) \circ \Pi^L(g,\overline{g}) = \Pi^L(c,d).$$

We conclude, that every left representation of a commutative group $\langle G, [\,]\rangle$ is a middle representation. Indeed,

$$\Pi^L(g,h) \circ \Pi^L(\overline{g},\overline{h}) = \Pi^L([g\, h\, \overline{g}],\overline{h}) = \Pi^L([g\, \overline{g}\, h],\overline{h}) = \Pi^L(h,\overline{h}) = \mathrm{id}_V$$

and

$$\Pi^L(g_1,g_2) \circ \Pi^L(g_3,g_4) \circ \Pi^L(g_5,g_6) = \Pi^L([[g_1g_2g_3]g_4g_5],g_6) = \Pi^L([[g_1g_3g_2]g_4g_5],g_6)$$
$$= \Pi^L([g_1g_3[g_2g_4g_5]],g_6) = \Pi^L([g_1g_3[g_5g_4g_2]],g_6) = \Pi^L([g_1g_3g_5],[g_4g_2g_6]) = \Pi^L([g_1g_3g_5],[g_6g_4g_2]).$$

Note that the converse holds only for the special kind of middle representations such that $\Pi^M(g,\overline{g}) = \mathrm{id}_V$. Therefore,

There is one-one correspondence between left representations of $\langle G, [\,]\rangle$ and binary representations of the retract $\mathrm{ret}_a(G, [\,])$.

Indeed, let $\Pi^L(g,a)$ be given, then $\rho(g) = \Pi^L(g,a)$ is such representation of the retract, as can be directly shown. Conversely, assume that $\rho(g)$ is a representation of the retract $\mathrm{ret}_a(G, [\,])$. Define $\Pi^L(g,h) = \rho(g) \circ \rho(\overline{h})^{-1}$, then $\Pi^L(g,h) \circ \Pi^L(u,u) = \rho(g) \circ \rho(\overline{h})^{-1} \circ \rho(u) \circ \rho(\overline{u})^{-1} = \rho(g \circledast (\overline{h})^{-1} \circ \circledast u) \circ \rho(\overline{u})^{-1} = \rho([\,[\,g\, a\, [\,\overline{a}\, h\, \overline{a}\,]\,]\, a\, u\,]) \circ \rho(\overline{u})^{-1} = \rho([\,g\, h\, g\,]) \circ \rho(\overline{u})^{-1} = \Pi^L([\,g\, h\, u\,], u)$,

MATRIX REPRESENTATIONS OF TERNARY GROUPS

Here we give several examples of matrix representations for concrete ternary groups. Let $G = \mathbb{Z}_3 \ni \{0,1,2\}$ and the ternary multiplication be $[ghu] = g - h + u$. Then $[ghu] = [uhg]$ and $\overline{0} = 0,\ \overline{1} = 1,\ \overline{2} = 2$, therefore $(G, [\,])$ is an idempotent medial ternary group. Thus $\Pi^L(g,h) = \Pi^R(h,g)$ and

$$\Pi^L(a,b) = \Pi^L(c,d) \Longleftrightarrow (a-b) = (c-d) \bmod 3. \tag{129}$$

The calculations give the left regular representation in the manifest matrix form

$$\Pi^L_{reg}(0,0) = \Pi^L_{reg}(2,2) = \Pi^L_{reg}(1,1) = \Pi^R_{reg}(0,0)$$
$$= \Pi^R_{reg}(2,2) = \Pi^R_{reg}(1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [1] \oplus [1] \oplus [1], \tag{130}$$

$$\Pi^L_{reg}(2,0) = \Pi^L_{reg}(1,2) = \Pi^L_{reg}(0,1) = \Pi^R_{reg}(2,1) = \Pi^R_{reg}(1,0) = \Pi^R_{reg}(0,2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
$$= [1] \oplus \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = [1] \oplus \left[-\frac{1}{2} + \frac{1}{2} i\sqrt{3}\right] \oplus \left[-\frac{1}{2} - \frac{1}{2} i\sqrt{3}\right], \tag{131}$$

$$\Pi^L_{reg}(2,1) = \Pi^L_{reg}(1,0) = \Pi^L_{reg}(0,2) = \Pi^R_{reg}(2,0) = \Pi^R_{reg}(1,2) = \Pi^R_{reg}(0,1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$= [1] \oplus \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = [1] \oplus \left[-\frac{1}{2} - \frac{1}{2} i\sqrt{3}\right] \oplus \left[-\frac{1}{2} + \frac{1}{2} i\sqrt{3}\right]. \tag{132}$$

Consider next the middle representation construction. The middle regular representation is defined by

$$\Pi^M_{reg}(g_1,g_2)\, t = \sum_{i=1}^{n} k_i \left[g_1 h_i g_2\right].$$

For regular representations we have

$$\Pi^M_{reg}(g_1,h_1)\circ\Pi^R_{reg}(g_2,h_2)=\Pi^R_{reg}(h_2,h_1)\circ\Pi^M_{reg}(g_1,g_2), \tag{133}$$
$$\Pi^M_{reg}(g_1,h_1)\circ\Pi^L_{reg}(g_2,h_2)=\Pi^L_{reg}(g_1,g_2)\circ\Pi^M_{reg}(h_2,h_1). \tag{134}$$

For the middle regular representation matrices we obtain

$$\Pi^M_{reg}(0,0)=\Pi^M_{reg}(1,2)=\Pi^M_{reg}(2,1)=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},$$
$$\Pi^M_{reg}(0,1)=\Pi^M_{reg}(1,0)=\Pi^M_{reg}(2,2)=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},$$
$$\Pi^M_{reg}(0,2)=\Pi^M_{reg}(2,0)=\Pi^M_{reg}(1,1)=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.$$

The above representation Π^M_{reg} of $\langle\mathbb{Z}_3,[\,]\rangle$ is equivalent to the orthogonal direct sum of two irreducible representations

$$\Pi^M_{reg}(0,0)=\Pi^M_{reg}(1,2)=\Pi^M_{reg}(2,1)=[1]\oplus\begin{bmatrix}-1&0\\0&1\end{bmatrix},$$
$$\Pi^M_{reg}(0,1)=\Pi^M_{reg}(1,0)=\Pi^M_{reg}(2,2)=[1]\oplus\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{bmatrix},$$
$$\Pi^M_{reg}(0,2)=\Pi^M_{reg}(2,0)=\Pi^M_{reg}(1,1)=[1]\oplus\begin{bmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{bmatrix},$$

i.e. one-dimensional trivial [1] and two-dimensional irreducible. Note, that in this example $\Pi^M(g,\bar{g})=\Pi^M(g,g)\neq\mathrm{id}_V$, but $\Pi^M(g,h)\circ\Pi^M(g,h)=\mathrm{id}_V$, and so Π^M are of the second degree.

Consider a more complicated example of left representations. Let $G=\mathbb{Z}_4\ni\{0,1,2,3\}$ and the ternary multiplication be

$$[ghu]=(g+h+u+1)\bmod 4. \tag{135}$$

We have the multiplication table

$$[g,h,0]=\begin{pmatrix}1&2&3&0\\2&3&0&1\\3&0&1&2\\0&1&2&3\end{pmatrix}\qquad[g,h,1]=\begin{pmatrix}2&3&0&1\\3&0&1&2\\0&1&2&3\\1&2&3&0\end{pmatrix}$$
$$[g,h,2]=\begin{pmatrix}3&0&1&2\\0&1&2&3\\1&2&3&0\\2&3&0&1\end{pmatrix}\qquad[g,h,3]=\begin{pmatrix}0&1&2&3\\1&2&3&0\\2&3&0&1\\3&0&1&2\end{pmatrix}$$

Then the skew elements are $\bar{0}=3,\ \bar{1}=2,\ \bar{2}=1,\ \bar{3}=0$, and therefore $(G,[\,])$ is a (non-idempotent) commutative ternary group. The left representation is defined by the expansion $\Pi^L_{reg}(g_1,g_2)\,t=\sum_{i=1}^n k_i\,[g_1g_2h_i]$, which means that (see the general formula (117))

$$\Pi^L_{reg}(g,h)\,|u>=|\,[ghu]>.$$

Analogously, for right and middle representations

$$\Pi^R_{reg}(g,h)\,|u>=|\,[ugh]>,\quad\Pi^M_{reg}(g,h)\,|u>=|\,[guh]>.$$

Therefore $|\,[ghu]>=|\,[ugh]>=|\,[guh]>$ and

$$\Pi^L_{reg}(g,h)=\Pi^R_{reg}(g,h)\,|u>=\Pi^M_{reg}(g,h)\,|u>,$$

so $\Pi^L_{reg}(g,h)=\Pi^R_{reg}(g,h)=\Pi^M_{reg}(g,h)$. Thus it is sufficient to consider the left representation only.

In this case the equivalence is $\Pi^L(a,b) = \Pi^L(c,d) \iff (a+b) = (c+d) \bmod 4$, and we obtain the following classes

$$\Pi^L_{reg}(0,0) = \Pi^L_{reg}(1,3) = \Pi^L_{reg}(2,2) = \Pi^L_{reg}(3,1) = \begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{pmatrix} = [1]\oplus[-1]\oplus[-i]\oplus[i],$$

$$\Pi^L_{reg}(0,1) = \Pi^L_{reg}(1,0) = \Pi^L_{reg}(2,3) = \Pi^L_{reg}(3,2) = \begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix} = [1]\oplus[-1]\oplus[-1]\oplus[-1],$$

$$\Pi^L_{reg}(0,2) = \Pi^L_{reg}(1,1) = \Pi^L_{reg}(2,0) = \Pi^L_{reg}(3,3) = \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0 \end{pmatrix} = [1]\oplus[-1]\oplus[i]\oplus[-i],$$

$$\Pi^L_{reg}(0,3) = \Pi^L_{reg}(1,2) = \Pi^L_{reg}(2,1) = \Pi^L_{reg}(3,0) = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} = [1]\oplus[-1]\oplus[1]\oplus[1].$$

It is seen that, due to the fact that the ternary operation (135) is commutative, there are only one-dimensional irreducible left representations.

Let us "algebralize" the above regular representations in the following way. From (118) we have, for the left representation

$$\Pi^L_{reg}(i,j) \circ \Pi^L_{reg}(k,l) = \Pi^L_{reg}(i,[jkl]), \tag{136}$$

where $[jkl] = j-k+l,\ i,j,k,l \in \mathbb{Z}_3$. Denote $\gamma^L_i = \Pi^L_{reg}(0,i),\ i \in \mathbb{Z}_3$, then we obtain the algebra with the relations

$$\gamma^L_i \gamma^L_j = \gamma^L_{i+j}. \tag{137}$$

Conversely, any matrix representation of $\gamma_i\gamma_j = \gamma_{i+j}$ leads to the left representation by $\Pi^L(i,j) = \gamma_{j-i}$. In the case of the middle regular representation we introduce $\gamma^M_{k+l} = \Pi^M_{reg}(k,l),\ k,l \in \mathbb{Z}_3$, then we obtain

$$\gamma^M_i \gamma^M_j \gamma^M_k = \gamma^M_{[ijk]}, \quad i,j,k \in \mathbb{Z}_3. \tag{138}$$

In some sense (138) can be treated as a `ternary analog of the Clifford algebra`. As before, any matrix representation of (138) gives the middle representation $\Pi^M(k,l) = \gamma_{k+l}$.

TERNARY ALGEBRAS AND HOPF ALGEBRAS

Let us consider associative ternary algebras [2, 115]. One can introduce an autodistributivity property $[[xyz]\,ab] = [[xab]\,[yab]\,[zab]]$ (see [72]). If we take 2 ternary operations $\{\ ,\ ,\ \}$ and $[\ ,\ ,\ ]$, then distributivity is given by $\{[xyz]\,ab\} = [\{xab\}\,\{yab\}\,\{zab\}]$. If $(+)$ is a binary operation (addition), then `left linearity` is

$$[(x+z), a, b] = [xab] + [zab]. \tag{139}$$

By analogy one can define central (middle) and right linearity. `Linearity` is defined, when left, middle and right linearity hold simultaneously.

An `associative ternary algebra` is a triple $\left(A, \mu_3, \eta^{(3)}\right)$, where A is a linear space over a field $\mathbb{K}$, μ_3 is a linear map $A\otimes A\otimes A \to A$ called `ternary multiplication` $\mu_3(a\otimes b\otimes c) = [abc]$ which is ternary associative $[[abc]\,de] = [a\,[bcd]\,e] = [ab\,[cde]]$ or

$$\mu_3 \circ (\mu_3 \otimes \mathrm{id} \otimes \mathrm{id}) = \mu_3 \circ (\mathrm{id} \otimes \mu_3 \otimes \mathrm{id}) = \mu_3 \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mu_3). \tag{140}$$

There are two types [45] of ternary unit maps $\eta^{(3)} : \mathbb{K} \to A$:

1) One `strong` unit map

$$\mu_3 \circ \left(\eta^{(3)} \otimes \eta^{(3)} \otimes \mathrm{id}\right) = \mu_3 \circ \left(\eta^{(3)} \otimes \mathrm{id} \otimes \eta^{(3)}\right) = \mu_3 \circ \left(\mathrm{id} \otimes \eta^{(3)} \otimes \eta^{(3)}\right) = \mathrm{id}; \tag{141}$$

2) Two `sequential` units $\eta^{(3)}_1$ and $\eta^{(3)}_2$ satisfying

$$\mu_3 \circ \left(\eta^{(3)}_1 \otimes \eta^{(3)}_2 \otimes \mathrm{id}\right) = \mu_3 \circ \left(\eta^{(3)}_1 \otimes \mathrm{id} \otimes \eta^{(3)}_2\right) = \mu_3 \circ \left(\mathrm{id} \otimes \eta^{(3)}_1 \otimes \eta^{(3)}_2\right) = \mathrm{id}; \tag{142}$$

In first case the ternary analog of the binary relation $\eta^{(2)}(x) = x1$, where $x \in \mathbb{K}$, $1 \in A$, is

$$\eta^{(3)}(x) = [x, 1, 1] = [1, 1, x] = [1, x, 1]. \tag{143}$$

Let (A, μ_A, η_A), (B, μ_B, η_B) and (C, μ_C, η_C) be ternary algebras, then the `ternary tensor product` space $A \otimes B \otimes C$ is naturally endowed with the structure of an algebra. The multiplication $\mu_{A\otimes B\otimes C}$ on $A \otimes B \otimes C$ reads

$$[(a_1 \otimes b_1 \otimes c_1)(a_2 \otimes b_2 \otimes c_2)(a_3 \otimes b_3 \otimes c_3)] = [a_1 a_2 a_3] \otimes [b_1 b_2 b_3] \otimes [c_1 c_2 c_3], \tag{144}$$

and so the set of ternary algebras is closed under taking ternary tensor products. A `ternary algebra map` (homomorphism) is a linear map between ternary algebras $f : A \to B$ which respects the ternary algebra structure

$$f([xyz]) = [f(x), f(y), f(z)], \tag{145}$$

$$f(1_A) = 1_B. \tag{146}$$

Let C be a linear space over a field $\mathbb{K}$.

A `ternary comultiplication` $\Delta^{(3)}$ is a linear map over a field $\mathbb{K}$ such that

$$\Delta_3 : C \to C \otimes C \otimes C. \tag{147}$$

In the standard Sweedler notations [37] $\Delta_3(a) = \sum_{i=1}^{n} a'_i \otimes a''_i \otimes a'''_i = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$. Consider different possible types of ternary coassociativity [45, 46].

1. A `standard` ternary coassociativity

$$(\Delta_3 \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta_3 = (\mathrm{id} \otimes \Delta_3 \otimes \mathrm{id}) \circ \Delta_3 = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta_3) \circ \Delta_3, \tag{148}$$

2. A `nonstandard` ternary Σ-coassociativity (Gluskin-type positional operatives)

$$(\Delta_3 \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta_3 = (\mathrm{id} \otimes (\sigma \circ \Delta_3) \otimes \mathrm{id}) \circ \Delta_3,$$

where $\sigma \circ \Delta_3(a) = \Delta_3(a) = a_{(\sigma(1))} \otimes a_{(\sigma(2))} \otimes a_{(\sigma(3))}$ and $\sigma \in \Sigma \subset S_3$.

3. A `permutational` ternary coassociativity

$$(\Delta_3 \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta_3 = \pi \circ (\mathrm{id} \otimes \Delta_3 \otimes \mathrm{id}) \circ \Delta_3,$$

where $\pi \in \Pi \subset S_5$.

A `ternary comediality` is

$$(\Delta_3 \otimes \Delta_3 \otimes \Delta_3) \circ \Delta_3 = \sigma_{medial} \circ (\Delta_3 \otimes \Delta_3 \otimes \Delta_3) \circ \Delta_3,$$

where $\sigma_{medial} = \binom{123456789}{147258369} \in S_9$. A `ternary counit` is defined as a map $\varepsilon^{(3)} : C \to \mathbb{K}$. In general, $\varepsilon^{(3)} \neq \varepsilon^{(2)}$ satisfying one of the conditions below. If Δ_3 is derived, then maybe $\varepsilon^{(3)} = \varepsilon^{(2)}$, but another counits may exist. There are two types of ternary counits:

1. Standard (`strong`) ternary counit

$$(\varepsilon^{(3)} \otimes \varepsilon^{(3)} \otimes \mathrm{id}) \circ \Delta_3 = (\varepsilon^{(3)} \otimes \mathrm{id} \otimes \varepsilon^{(3)}) \circ \Delta_3 = (\mathrm{id} \otimes \varepsilon^{(3)} \otimes \varepsilon^{(3)}) \circ \Delta_3 = \mathrm{id}, \tag{149}$$

2. Two `sequential` (polyadic) counits $\varepsilon_1^{(3)}$ and $\varepsilon_2^{(3)}$

$$(\varepsilon_1^{(3)} \otimes \varepsilon_2^{(3)} \otimes \mathrm{id}) \circ \Delta = (\varepsilon_1^{(3)} \otimes \mathrm{id} \otimes \varepsilon_2^{(3)}) \circ \Delta = (\mathrm{id} \otimes \varepsilon_1^{(3)} \otimes \varepsilon_2^{(3)}) \circ \Delta = \mathrm{id}, \tag{150}$$

Below we will consider only the first standard type of associativity (148). The σ-`cocommutativity` is defined as $\sigma \circ \Delta_3 = \Delta_3$.

A `ternary coalgebra` is a triple $(C, \Delta_3, \varepsilon^{(3)})$, where C is a linear space and Δ_3 is a ternary comultiplication (147) which is coassociative in one of the above senses and $\varepsilon^{(3)}$ is one of the above counits.

Let $(A, \mu^{(3)})$ be a ternary algebra and (C, Δ_3) be a ternary coalgebra and $f, g, h \in \mathrm{Hom}_{\mathbb{K}}(C, A)$. `Ternary convolution product` is

$$[f, g, h]_* = \mu^{(3)} \circ (f \otimes g \otimes h) \circ \Delta_3 \tag{151}$$

or in the Sweedler notation $[f, g, h]_*(a) = [f(a_{(1)})\, g(a_{(2)})\, h(a_{(3)})]$.

A ternary coalgebra is called `derived`, if there exists a binary (usual, see e.g. [37]) coalgebra $\Delta_2 : C \to C \otimes C$ such that

$$\Delta_{3,\mathrm{der}} = (\mathrm{id} \otimes \Delta_2) \otimes \Delta_2. \quad (152)$$

A ternary bialgebra B is $\left(B, \mu^{(3)}, \eta^{(3)}, \Delta_3, \varepsilon^{(3)}\right)$ for which $\left(B, \mu^{(3)}, \eta^{(3)}\right)$ is a ternary algebra and $\left(B, \Delta_3, \varepsilon^{(3)}\right)$ is a ternary coalgebra and they are compatible

$$\Delta_3 \circ \mu^{(3)} = \mu^{(3)} \circ \Delta_3 \quad (153)$$

One can distinguish four kinds of ternary bialgebras with respect to a "being derived" property:

1. A Δ-`derived` ternary bialgebra
$$\Delta_3 = \Delta_{3,\mathrm{der}} = (\mathrm{id} \otimes \Delta_2) \circ \Delta_2 \quad (154)$$
2. A μ-`derived` ternary bialgebra
$$\mu^{(3)}_{der} = \mu^{(3)}_{der} = \mu^{(2)} \circ \left(\mu^{(2)} \otimes \mathrm{id}\right) \quad (155)$$
3. A `derived` ternary bialgebra is simultaneously μ-derived and Δ-derived ternary bialgebra.
4. A `non-derived` ternary bialgebra which does not satisfy (154) and (155).

Possible types of ternary antipodes can be defined by analogy with binary coalgebras.

A `skew ternary antipod` is

$$\mu^{(3)} \circ (S^{(3)}_{skew} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta_3 = \mu^{(3)} \circ (\mathrm{id} \otimes S^{(3)}_{skew} \otimes \mathrm{id}) \circ \Delta_3 = \mu^{(3)} \circ (\mathrm{id} \otimes \mathrm{id} \otimes S^{(3)}_{skew}) \circ \Delta_3 = \mathrm{id}\,. \quad (156)$$

If only one equality from (156) is satisfied, the corresponding skew antipode is called `left`, `middle` or `right`.

`Strong ternary antipode` is

$$\left(\mu^{(2)} \otimes \mathrm{id}\right) \circ (\mathrm{id} \otimes S^{(3)}_{strong} \otimes \mathrm{id}) \circ \Delta_3 = 1 \otimes \mathrm{id}, \left(\mathrm{id} \otimes \mu^{(2)}\right) \circ (\mathrm{id} \otimes \mathrm{id} \otimes S^{(3)}_{strong}) \circ \Delta_3 = \mathrm{id} \otimes 1,$$

where 1 is a unit of algebra.

If in a ternary coalgebra the relation

$$\Delta_3 \circ S = \tau_{13} \circ (S \otimes S \otimes S) \circ \Delta_3 \quad (157)$$

holds true, where $\tau_{13} = \binom{123}{321}$, then it is called `skew-involutive`.

A ternary Hopf algebra $\left(H, \mu^{(3)}, \eta^{(3)}, \Delta_3, \varepsilon^{(3)}, S^{(3)}\right)$ is a ternary bialgebra with a ternary antipode $S^{(3)}$ of the corresponding above type .

Let us consider concrete constructions of ternary comultiplications, bialgebras and Hopf algebras. A `ternary group-like element` can be defined by $\Delta_3(g) = g \otimes g \otimes g$, and for 3 such elements we have

$$\Delta_3([g_1 g_2 g_3]) = \Delta_3(g_1)\, \Delta_3(g_2)\, \Delta_3(g_3)\,. \quad (158)$$

But an analog of the binary primitive element (satisfying $\Delta^{(2)}(x) = x \otimes 1 + 1 \otimes x$) cannot be chosen simply as $\Delta_3(x) = x \otimes e \otimes e + e \otimes x \otimes e + e \otimes e \otimes x$, since the algebra structure is not preserved. Nevertheless, if we introduce `two` idempotent units e_1, e_2 satisfying "semiorthogonality" $[e_1 e_1 e_2] = 0$, $[e_2 e_2 e_1] = 0$, then

$$\Delta_3(x) = x \otimes e_1 \otimes e_2 + e_2 \otimes x \otimes e_1 + e_1 \otimes e_2 \otimes x \quad (159)$$

and now $\Delta_3([x_1 x_2 x_3]) = [\Delta_3(x_1)\, \Delta_3(x_2)\, \Delta_3(x_3)]$. Using (159) $\varepsilon(x) = 0$, $\varepsilon(e_{1,2}) = 1$, and $S^{(3)}(x) = -x$, $S^{(3)}(e_{1,2}) = e_{1,2}$, one can construct a ternary universal enveloping algebra in full analogy with the binary case (see e.g. [39]).

One of the most important examples of noncocommutative Hopf algebras is the well known Sweedler Hopf algebra [37] which in the binary case has two generators x and y satisfying

$$\mu^{(2)}(x, x) = 1, \quad (160)$$
$$\mu^{(2)}(y, y) = 0, \quad (161)$$
$$\sigma^{(2)}_{+}(xy) = -\sigma^{(2)}_{-}(xy)\,. \quad (162)$$

It has the following comultiplication

$$\Delta_2(x) = x \otimes x, \quad (163)$$
$$\Delta_2(y) = y \otimes x + 1 \otimes y, \quad (164)$$

counit $\varepsilon^{(2)}(x) = 1$, $\varepsilon^{(2)}(y) = 0$, and antipod $S^{(2)}(x) = x$, $S^{(2)}(y) = -y$, which respect the algebra structure. In the derived case a `ternary Sweedler algebra` is generated also by two generators x and y obeying

$$\mu^{(3)}(x,e,x) = \mu^{(3)}(e,x,x) = \mu^{(3)}(x,x,e) = e, \tag{165}$$

$$\sigma_+^{(3)}([yey]) = 0, \tag{166}$$

$$\sigma_+^{(3)}([xey]) = -\sigma_-^{(3)}([xey]). \tag{167}$$

The derived Hopf algebra structure is given by

$$\Delta_3(x) = x \otimes x \otimes x, \tag{168}$$

$$\Delta_3(y) = y \otimes x \otimes x + e \otimes y \otimes x + e \otimes e \otimes y, \tag{169}$$

$$\varepsilon^{(3)}(x) = \varepsilon^{(2)}(x) = 1, \tag{170}$$

$$\varepsilon^{(3)}(y) = \varepsilon^{(2)}(y) = 0, \tag{171}$$

$$S^{(3)}(x) = S^{(2)}(x) = x, \tag{172}$$

$$S^{(3)}(y) = S^{(2)}(y) = -y, \tag{173}$$

and it can be checked that (168)-(170) are algebra maps, while (172) are antialgebra maps. To obtain a non-derived ternary Sweedler example we have the following possibilities: 1) one "even" generator x, two "odd" generators $y_{1,2}$ and one ternary unit e; 2) two "even" generators $x_{1,2}$, one "odd" generator y and two ternary units $e_{1,2}$. In the first case the ternary algebra structure is (no summation, $i = 1, 2$)

$$[xxx] = e, \tag{174}$$

$$[y_i y_i y_i] = 0, \tag{175}$$

$$\sigma_+^{(3)}([y_i x y_i]) = \sigma_+^{(3)}([x y_i x]) = 0, \tag{176}$$

$$[xey_i] = -[xy_ie],$$

$$[exy_i] = -[y_ixe], \tag{177}$$

$$[ey_ix] = -[y_iex], \tag{178}$$

$$\sigma_+^{(3)}([y_1xy_2]) = -\sigma_-^{(3)}([y_1xy_2]). \tag{179}$$

The corresponding ternary Hopf algebra structure is

$$\Delta_3(x) = x \otimes x \otimes x,\ \Delta_3(y_{1,2}) = y_{1,2} \otimes x \otimes x + e_{1,2} \otimes y_{2,1} \otimes x + e_{1,2} \otimes e_{2,1} \otimes y_{2,1}, \tag{180}$$

$$\varepsilon^{(3)}(x) = 1,\ \ \varepsilon^{(3)}(y_i) = 0, \tag{181}$$

$$S^{(3)}(x) = x, \quad S^{(3)}(y_i) = -y_i. \tag{182}$$

In the second case we have for the algebra structure

$$[x_i x_j x_k] = \delta_{ij}\delta_{ik}\delta_{jk} e_i, \quad [yyy] = 0, \tag{183}$$

$$\sigma_+^{(3)}([yx_iy]) = 0,\ \sigma_+^{(3)}([x_iyx_i]) = 0, \tag{184}$$

$$\sigma_+^{(3)}([y_1xy_2]) = 0, \quad \sigma_-^{(3)}([y_1xy_2]) = 0, \tag{185}$$

and the ternary Hopf algebra structure is

$$\Delta_3(x_i) = x_i \otimes x_i \otimes x_i,$$

$$\Delta_3(y) = y \otimes x_1 \otimes x_1 + e_1 \otimes y \otimes x_2 + e_1 \otimes e_2 \otimes y, \tag{186}$$

$$\varepsilon^{(3)}(x_i) = 1, \tag{187}$$

$$\varepsilon^{(3)}(y) = 0, \tag{188}$$

$$S^{(3)}(x_i) = x_i, \tag{189}$$

$$S^{(3)}(y) = -y. \tag{190}$$

TERNARY QUANTUM GROUPS

A ternary commutator can be obtained in different ways [116]. We will consider the simplest version called a Nambu bracket (see e.g. [2, 30]). Let us introduce two maps $\omega_\pm^{(3)} : A \otimes A \otimes A \to A \otimes A \otimes A$ by

$$\omega_+^{(3)}(a \otimes b \otimes c) = a \otimes b \otimes c + b \otimes c \otimes a + c \otimes a \otimes b, \tag{191}$$

$$\omega_-^{(3)}(a \otimes b \otimes c) = b \otimes a \otimes c + c \otimes b \otimes a + a \otimes c \otimes b. \tag{192}$$

Thus, obviously $\mu^{(3)} \circ \omega_{\pm}^{(3)} = \sigma_{\pm}^{(3)} \circ \mu^{(3)}$, where $\sigma_{\pm}^{(3)} \in S_3$ denotes a sum of terms having even and odd permutations respectively. In the binary case $\omega_{+}^{(2)} = \mathrm{id} \otimes \mathrm{id}$ and $\omega_{-}^{(2)} = \tau$ is the twist operator $\tau : a \otimes b \to b \otimes a$, while $\mu^{(2)} \circ \omega_{-}^{(2)}$ is permutation $\sigma_{-}^{(2)} (ab) = ba$. So the Nambu product is $\omega_{N}^{(3)} = \omega_{+}^{(3)} - \omega_{-}^{(3)}$, and the ternary commutator is $[,,]_N = \sigma_N^{(3)} = \sigma_{+}^{(3)} - \sigma_{-}^{(3)}$, or [30]

$$[a, b, c]_N = [abc] + [bca] + [cab] - [cba] - [acb] - [bac] \tag{193}$$

An `abelian ternary algebra` is defined by the vanishing of the Nambu bracket $[a, b, c]_N = 0$ or `ternary commutation` relation $\sigma_{+}^{(3)} = \sigma_{-}^{(3)}$. By analogy with the binary case a `deformed ternary algebra` can be defined by

$$\sigma_{+}^{(3)} = q\sigma_{-}^{(3)} \text{ or } [abc] + [bca] + [cab] = q\left([cba] + [acb] + [bac]\right), \tag{194}$$

where multiplication by q is treated as an external operation.

Let us consider a ternary analog of the Woronowicz example of a bialgebra construction, which in the binary case has two generators satisfying $xy = qyx$ (or $\sigma_{+}^{(2)} (xy) = q\sigma_{-}^{(2)} (xy)$), then the following coproducts

$$\Delta_2 (x) = x \otimes x \tag{195}$$

$$\Delta_2 (y) = y \otimes x + 1 \otimes y \tag{196}$$

are algebra maps. In the derived ternary case using (194) we have

$$\sigma_{+}^{(3)} ([xey]) = q\sigma_{-}^{(3)} ([xey]), \tag{197}$$

where e is the ternary unit and ternary coproducts are

$$\Delta_3 (e) = e \otimes e \otimes e, \tag{198}$$

$$\Delta_3 (x) = x \otimes x \otimes x, \tag{199}$$

$$\Delta_3 (y) = y \otimes x \otimes x + e \otimes y \otimes x + e \otimes e \otimes y, \tag{200}$$

which are `ternary algebra maps`, i.e. they satisfy

$$\sigma_{+}^{(3)} ([\Delta_3 (x) \Delta_3 (e) \Delta_3 (y)]) = q\sigma_{-}^{(3)} ([\Delta_3 (x) \Delta_3 (e) \Delta_3 (y)]). \tag{201}$$

Let us consider the group $G = SL(n, \mathbb{K})$. Then the algebra generated by $a_j^i \in SL(n, \mathbb{K})$ can be endowed with the structure of a ternary Hopf algebra (see e.g. [117] for the binary case) by choosing the ternary coproduct, counit and antipode as (here summation is implied)

$$\Delta_3 \left(a_j^i\right) = a_k^i \otimes a_l^k \otimes a_j^l, \quad \varepsilon \left(a_j^i\right) = \delta_j^i, \quad S^{(3)} \left(a_j^i\right) = \left(a^{-1}\right)_j^i. \tag{202}$$

This antipode is a skew one since from (156) it follows that

$$\mu^{(3)} \circ (S^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta_3 \left(a_j^i\right) = S^{(3)} \left(a_k^i\right) a_l^k a_j^l = \left(a^{-1}\right)_k^i a_l^k a_j^l = \delta_l^i a_j^l = a_j^i. \tag{203}$$

This ternary Hopf algebra is derived since for $\Delta^{(2)} = a_j^i \otimes a_k^j$ we have

$$\Delta_3 = \left(\mathrm{id} \otimes \Delta^{(2)}\right) \otimes \Delta^{(2)} \left(a_j^i\right) = \left(\mathrm{id} \otimes \Delta^{(2)}\right) \left(a_k^i \otimes a_j^k\right) = a_k^i \otimes \Delta^{(2)} \left(a_j^k\right) = a_k^i \otimes a_l^k \otimes a_j^l. \tag{204}$$

In the most important case $n = 2$ we can obtain the manifest action of the ternary coproduct Δ_3 on components. Possible non-derived matrix representations of the ternary product can be done only by four-rank $n \times n \times n \times n$ twice covariant and twice contravariant tensors $\left\{a_{kl}^{ij}\right\}$. Among all products the non-derived ones are only the following: $a_{jk}^{oi} b_{oo}^{jl} c_{il}^{ko}$ and $a_{ok}^{ij} b_{io}^{ol} c_{il}^{ko}$ (where o is any index). So using e.g. the first choice we can define the non-derived Hopf algebra structure by

$$\Delta_3 \left(a_{kl}^{ij}\right) = a_{v\rho}^{i\mu} \otimes a_{kl}^{v\sigma} \otimes a_{\mu\sigma}^{\rho j}, \tag{205}$$

$$\varepsilon \left(a_{kl}^{ij}\right) = \frac{1}{2} \left(\delta_k^i \delta_l^j + \delta_l^i \delta_k^j\right), \tag{206}$$

and the skew antipod $s_{kl}^{ij} = S^{(3)} \left(a_{kl}^{ij}\right)$ which is a solution of the equation $s_{v\rho}^{i\mu} a_{kl}^{v\sigma} = \delta_\rho^i \delta_k^\mu \delta_l^\sigma$.

Next consider ternary dual pair $k(G)$ (push-forward) and $\mathcal{F}(G)$ (pull-back) which are related by $k^*(G) \cong \mathcal{F}(G)$ (see e.g. [118]). Here $k(G) = \operatorname{span}(G)$ is a ternary group algebra (G has a ternary product $[\,]_G$ or $\mu_G^{(3)}$) over a field k. If $u \in k(G)$ ($u = u^i x_i, x_i \in G$), then

$$[uvw]_k = u^i v^j w^l [x_i x_j x_l]_G \tag{207}$$

is associative, and so $(k(G), [\,]_k)$ becomes a ternary algebra. Define a ternary coproduct $\Delta_3 : k(G) \to k(G) \otimes k(G) \otimes k(G)$ by

$$\Delta_3(u) = u^i x_i \otimes x_i \otimes x_i \tag{208}$$

(derived and associative), then $\Delta_3([uvw]_k) = [\Delta_3(u)\Delta_3(v)\Delta_3(w)]_k$, and $k(G)$ is a ternary bialgebra. If we define a ternary antipod by $S_k^{(3)} = u^i \bar{x}_i$, where $\bar{x}_i$ is a skew element of x_i, then $k(G)$ becomes a ternary Hopf algebra.

In the dual case of functions $\mathcal{F}(G) : \{\varphi : G \to k\}$ a ternary product $[\,]_{\mathcal{F}}$ or $\mu_{\mathcal{F}}^{(3)}$ (derived and associative) acts on $\psi(x, y, z)$ as

$$\left(\mu_{\mathcal{F}}^{(3)} \psi\right)(x) = \psi(x, x, x), \tag{209}$$

and so $\mathcal{F}(G)$ is a ternary algebra. Let $\mathcal{F}(G) \otimes \mathcal{F}(G) \otimes \mathcal{F}(G) \cong \mathcal{F}(G \times G \times G)$, then we define a ternary coproduct $\Delta_3 : \mathcal{F}(G) \to \mathcal{F}(G) \otimes \mathcal{F}(G) \otimes \mathcal{F}(G)$ as

$$(\Delta_3 \varphi)(x, y, z) = \varphi([xyz]_{\mathcal{F}}), \tag{210}$$

which is derive and associative. Thus we can obtain $\Delta_3([\varphi_1\varphi_2\varphi_3]_{\mathcal{F}}) = [\Delta_3(\varphi_1)\Delta_3(\varphi_2)\Delta_3(\varphi_3)]_{\mathcal{F}}$, and therefore $\mathcal{F}(G)$ is a ternary bialgebra. If we define a ternary antipod by

$$S_{\mathcal{F}}^{(3)}(\varphi) = \varphi(\bar{x}), \tag{211}$$

where $\bar{x}$ is a skew element of x, then $\mathcal{F}(G)$ becomes a ternary Hopf algebra.

Let us introduce a ternary analog of the R-matrix. For a ternary Hopf algebra H we consider a linear map $R^{(3)} : H \otimes H \otimes H \to H \otimes H \otimes H$.

A ternary Hopf algebra $\left(H, \mu^{(3)}, \eta^{(3)}, \Delta_3, \varepsilon^{(3)}, S^{(3)}\right)$ is called `quasifiveangular`[4] if it satisfies

$$(\Delta_3 \otimes \mathrm{id} \otimes \mathrm{id}) = R_{145}^{(3)} R_{245}^{(3)} R_{345}^{(3)}, \tag{212}$$

$$(\mathrm{id} \otimes \Delta_3 \otimes \mathrm{id}) = R_{125}^{(3)} R_{145}^{(3)} R_{135}^{(3)}, \tag{213}$$

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta_3) = R_{125}^{(3)} R_{124}^{(3)} R_{123}^{(3)}, \tag{214}$$

where as usual the index of R denotes action component positions.

Using the standard procedure (see e.g. [39, 119, 120]) we obtain a set of abstract `ternary quantum Yang-Baxter equations`, one of which has the form

$$R_{243}^{(3)} R_{342}^{(3)} R_{125}^{(3)} R_{145}^{(3)} R_{135}^{(3)} = R_{123}^{(3)} R_{132}^{(3)} R_{145}^{(3)} R_{245}^{(3)} R_{345}^{(3)}, \tag{215}$$

and others can be obtained by corresponding permutations. The classical ternary Yang-Baxter equations form a one parameter family of solutions $R(t)$ can be obtained by the expansion

$$R^{(3)}(t) = e \otimes e \otimes e + rt + \mathcal{O}\left(t^2\right), \tag{216}$$

where r is a ternary classical R-matrix, then e.g. for (215) we have

$$\begin{aligned} & r_{342}r_{125}r_{145}r_{135} + r_{243}r_{125}r_{145}r_{135} + r_{243}r_{342}r_{145}r_{135} + r_{243}r_{342}r_{125}r_{135} + r_{243}r_{342}r_{125}r_{145} \\ & = r_{132}r_{145}r_{245}r_{345} + r_{123}r_{145}r_{245}r_{345} + r_{123}r_{132}r_{245}r_{345} + r_{123}r_{132}r_{145}r_{345} + r_{123}r_{132}r_{145}r_{245}. \end{aligned}$$

For three ternary Hopf algebras $\left(H_{A,B,C}, \mu_{A,B,C}^{(3)}, \eta_{A,B,C}^{(3)}, \Delta_{A,B,C}^{(3)}, \varepsilon_{A,B,C}^{(3)}, S_{A,B,C}^{(3)}\right)$ we can introduce a non-degenerate ternary "pairing" (see e.g. [119] for the binary case) $\langle\ ,\ ,\ \rangle^{(3)} : H_A \times H_B \times H_C \to \mathbb{K}$, trilinear over $\mathbb{K}$, satisfying

$$\begin{aligned} &\left\langle \eta_A^{(3)}(a), b, c \right\rangle^{(3)} = \left\langle a, \varepsilon_B^{(3)}(b), c \right\rangle^{(3)}, \quad \left\langle a, \eta_B^{(3)}(b), c \right\rangle^{(3)} = \left\langle \varepsilon_A^{(3)}(a), b, c \right\rangle^{(3)}, \\ &\left\langle b, \eta_B^{(3)}(b), c \right\rangle^{(3)} = \left\langle a, b, \varepsilon_C^{(3)}(c) \right\rangle^{(3)}, \quad \left\langle a, b, \eta_C^{(3)}(c) \right\rangle^{(3)} = \left\langle a, \varepsilon_B^{(3)}(b), c \right\rangle^{(3)}, \\ &\left\langle a, b, \eta_C^{(3)}(c) \right\rangle^{(3)} = \left\langle \varepsilon_A^{(3)}(a), b, c \right\rangle^{(3)}, \quad \left\langle \eta_A^{(3)}(a), b, c \right\rangle^{(3)} = \left\langle a, b, \varepsilon_C^{(3)}(c) \right\rangle^{(3)}, \end{aligned}$$

[4] The reason for such notation is clear from (215).

$$\left\langle \mu_A^{(3)}\left(a_1\otimes a_2\otimes a_3\right),b,c\right\rangle^{(3)}=\left\langle a_1\otimes a_2\otimes a_3,\Delta_B^{(3)}\left(b\right),c\right\rangle^{(3)},$$
$$\left\langle \Delta_A^{(3)}\left(a\right),b_1\otimes b_2\otimes b_3,c\right\rangle^{(3)}=\left\langle a,\mu_B^{(3)}\left(b_1\otimes b_2\otimes b_3\right),c\right\rangle^{(3)},$$
$$\left\langle a,\mu_B^{(3)}\left(b_1\otimes b_2\otimes b_3\right),c\right\rangle^{(3)}=\left\langle a,b_1\otimes b_2\otimes b_3,\Delta_C^{(3)}\left(c\right)\right\rangle^{(3)},$$
$$\left\langle a,\Delta_B^{(3)}\left(b\right),c_1\otimes c_2\otimes c_3\right\rangle^{(3)}=\left\langle a,b,\mu_C^{(3)}\left(c_1\otimes c_2\otimes c_3\right)\right\rangle^{(3)},$$
$$\left\langle a,b,\mu_C^{(3)}\left(c_1\otimes c_2\otimes c_3\right)\right\rangle^{(3)}=\left\langle \Delta_A^{(3)}\left(a\right),b,c_1\otimes c_2\otimes c_3\right\rangle^{(3)},$$
$$\left\langle a_1\otimes a_2\otimes a_3,b,\Delta_C^{(3)}\left(c\right)\right\rangle^{(3)}=\left\langle \mu_A^{(3)}\left(a_1\otimes a_2\otimes a_3\right),b,c\right\rangle^{(3)},$$

$$\left\langle S_A^{(3)}\left(a\right),b,c\right\rangle^{(3)}=\left\langle a,S_B^{(3)}\left(b\right),c\right\rangle^{(3)}=\left\langle a,b,S_C^{(3)}\left(c\right)\right\rangle^{(3)},$$

where $a,a_i\in H_A$, $b,b_i\in H_B$. The ternary “paring” between $H_A\otimes H_A\otimes H_A$ and $H_B\otimes H_B\otimes H_B$ is given by $\langle a_1\otimes a_2\otimes a_3,b_1\otimes b_2\otimes b_3\rangle^{(3)}=\langle a_1,b_1\rangle^{(3)}\langle a_2,b_2\rangle^{(3)}\langle a_3,b_3\rangle^{(3)}$. These constructions can naturally lead to ternary generalizations of the duality concept and the quantum double which are key ingredients in the theory of quantum groups [39, 120, 121].

CONCLUSIONS

In this paper we have presented a review of polyadic systems and their representations, ternary algebras and Hopf algebras. We have classified general polyadic systems and considered their homomorphisms and their multiplace generalizations, paying attention to their associativity. We have defined multiplace representations and multiactions and have given examples of matrix representations for some ternary groups. We have defined and investigated ternary algebras and Hopf algebras, and have given some examples. We then considered some ternary generalizations of quantum groups and the Yang-Baxter equation.

REFERENCES

1. Kerner R. Ternary algebraic structures and their applications in physics // Paris, 2000. - 15 p. (Preprint Univ. P. & M. Curie).
2. de Azcarraga J. A., Izquierdo J. M. n-ary algebras: a review with applications // J. Phys. - 2010. - Vol. A43. - P. 293001.
3. Dörnte W. Unterschungen über einen verallgemeinerten Gruppenbegriff // Math. Z. - 1929. - Vol. 29. - P. 1–19.
4. Prüfer H. Theorie der abelshen Gruppen I. Grundeigenschaften // Math. Z. - 1924. - Vol. 20. - P. 165–187.
5. Post E. L. Polyadic groups // Trans. Amer. Math. Soc. - 1940. - Vol. 48. - P. 208–350.
6. Hosszú L. M. On the explicit form of n-group operation // Publ. Math. Debrecen. - 1963. - Vol. 10. - P. 88–92.
7. Gluskin L. M. Position operatives // Math. Sbornik. - 1965. - Vol. 68(110). - № 3. - P. 444–472.
8. Dudek W. A., Głazek K., Gleichgewicht B. A note on the axioms of n-groups // Coll. Math. Soc. J. Bolyai. 29. Universal Algebra. - Esztergom (Hungary). , 1977. - P. 195–202.
9. Rusakov S. A. A definition of an n-ary group // Dokl. Akad. Nauk BSSR. - 1979. - Vol. 23. - P. 965–967.
10. Weyl H. Classical Groups, Their Invariants and Representations. - Princeton: Princeton Univ. Press, 1946.
11. Fulton W., Harris J. Representation Theory: a First Course. - N. Y.: Springer, 1991.
12. Cornwell J. F. Group Theory in Physics: An Introduction. - London: Academic Press, 1997. - 349 p.
13. Curtis C. W., Reiner I. Representation theory of finite groups and associative algebras. - Providence: AMS, 1962.
14. Collins M. J. Representations And Characters Of Finite Groups. - Cambridge: Cambridge University Press, 1990. - 242 p.
15. Kapranov M., Gelfand I. M., Zelevinskii A. Discriminants, Resultants and Multidimensional Determinants. - Berlin: Birkhäuser, 1994. - 234 p.
16. Sokolov N. P. Introduction to the Theory of Multidimensional Matrices. - Kiev: Naukova Dumka, 1972. - 175 p.
17. Kawamura Y. Cubic matrices, generalized spin algebra and uncertainty relation // Progr. Theor. Phys. - 2003. - Vol. 110. - P. 579–587.
18. Rausch de Traubenberg M. Cubic extentions of the poincare algebra // Phys. Atom. Nucl. - 2008. - Vol. 71. - P. 1102–1108.
19. Borowiec A., Dudek W., Duplij S. Bi-element representations of ternary groups // Comm. Algebra. - 2006. - Vol. 34. - № 5. - P. 1651–1670.
20. Dudek W. A., Shahryari M. Representation theory of polyadic groups // Algebr. Represent. Theory. - 2012. - Vol. 15. - № 1. - P. 29–51.
21. Lõhmus J., Paal E., Sorgsepp L. Nonassociative Algebras in Physics. - Palm Harbor: Hadronic Press, 1994. - 271 p.
22. Lounesto P., Ablamowicz R. Clifford Algebras: Applications To Mathematics, Physics, And Engineering. - Birkhäuser, 2004. - 626 p.
23. Georgi H. Lie Algebras in Particle Physics. - New York: Perseus Books, 1999. - 320 p.

24. Abramov V. $\mathbb{Z}_3$-graded analogues of Clifford algebras and algebra of $\mathbb{Z}_3$-graded symmetries // Algebras Groups Geom. - 1995. - Vol. 12. - № 3. - P. 201–221.
25. Abramov V. Ternary generalizations of Grassmann algebra // Proc. Est. Acad. Sci., Phys. Math. - 1996. - Vol. 45. - № 2-3. - P. 152–160.
26. Abramov V., Kerner R., Le Roy B. Hypersymmetry: a $\mathbb{Z}_3$-graded generalization of supersymmetry // J. Math. Phys. - 1997. - Vol. 38. - P. 1650–1669.
27. Filippov V. T. n-Lie algebras // Sib. Math. J. - 1985. - Vol. 26. - P. 879–891.
28. Michor P. W., Vinogradov A. M. n-ary Lie and associative algebras // Rend. Sem. Mat. Univ. Pol. Torino. - 1996. - Vol. 54. - № 4. - P. 373–392.
29. Nambu Y. Generalized Hamiltonian dynamics // Phys. Rev. - 1973. - Vol. 7. - P. 2405–2412.
30. Takhtajan L. On foundation of the generalized Nambu mechanics // Commun. Math. Phys. - 1994. - Vol. 160. - P. 295–315.
31. Bagger J., Lambert N. Comments on multiple M2-branes // - 2008. - Vol. 2. - P. 105.
32. Bagger J., Lambert N. Gauge symmetry and supersymmetry of multiple M2-branes // Phys. Rev. - 2008. - Vol. D77. - P. 065008.
33. Gustavsson A. One-loop corrections to Bagger-Lambert theory // Nucl. Phys. - 2009. - Vol. B807. - P. 315–333.
34. Ho P.-M., Hou R.-C., Matsuo Y., Shiba S. M5-brane in three-form flux and multiple M2-branes // - 2008. - Vol. 08. - P. 014.
35. Low A. M. Worldvolume superalgebra of BLG theory with Nambu-Poisson structure // - 2010. - Vol. 04. - P. 089.
36. Abe E. Hopf Algebras. - Cambridge: Cambridge Univ. Press, 1980. - 221 p.
37. Sweedler M. E. Hopf Algebras. - New York: Benjamin, 1969. - 336 p.
38. Montgomery S. Hopf algebras and their actions on rings. - Providence: AMS, 1993. - 238 p.
39. Kassel C. Quantum Groups. - New York: Springer-Verlag, 1995. - 531 p.
40. Shnider S., Sternberg S. Quantum Groups. - Boston: International Press, 1993. - 371 p.
41. Duplij S., Li F. Regular solutions of quantum Yang-Baxter equation from weak Hopf algebras // Czech. J. Phys. - 2001. - Vol. 51. - № 12. - P. 1306–1311.
42. Duplij S., Sinel'shchikov S. Quantum enveloping algebras with von Neumann regular Cartan-like generators and the Pierce decomposition // Commun. Math. Phys. - 2009. - Vol. 287. - № 1. - P. 769–785.
43. Li F., Duplij S. Weak Hopf algebras and singular solutions of quantum Yang-Baxter equation // Commun. Math. Phys. - 2002. - Vol. 225. - № 1. - P. 191–217.
44. Duplij S., Sinel'shchikov S. Classification of $U_q\,(\mathrm{sl}_2)$-module algebra structures on the quantum plane // J. Math. Physics, Analysis, Geometry. - 2010. - Vol. 6. - № 6. - P. 21–46.
45. Duplij S. Ternary Hopf algebras // Symmetry in Nonlinear Mathematical Physics. - Kiev. Institute of Mathematics, 2001. - P. 25-34.
46. Borowiec A., Dudek W., Duplij S. Basic concepts of ternary Hopf algebras // Journal of Kharkov National University, ser. Nuclei, Particles and Fields. - 2001. - Vol. 529. - № 3(15). - P. 21–29.
47. Bergman G. M. An invitation to general algebra and universal constructions. - Berkeley: University of California, 1995. - 358 p.
48. Hausmann B. A., Ore Ø. Theory of quasigroups // Amer. J. Math. - 1937. - Vol. 59. - P. 983–1004.
49. Clifford A. H., Preston G. B. The Algebraic Theory of Semigroups. Vol. 1 - Providence: Amer. Math. Soc., 1961.
50. Brandt H. Über eine Verallgemeinerung des Gruppenbegriffes // Math. Annalen. - 1927. - Vol. 96. - P. 360–367.
51. Bruck R. H. A Survey on Binary Systems. - New York: Springer-Verlag, 1966.
52. Bourbaki N. Elements of Mathematics: Algebra 1. - Springer, 1998.
53. Dudek W. A. Remarks to głazek's results on n-ary groups // Discuss. Math., Gen. Algebra Appl. - 2007. - Vol. 27. - № 2. - P. 199–233.
54. Dudek W. A., Michalski J. On retract of polyadic groups // Demonstratio Math. - 1984. - Vol. 17. - P. 281–301.
55. Thurston H. A. Partly associative operations // J. London Math. Soc. - 1949. - Vol. 24. - P. 260–271.
56. Belousov V. D. n-ary Quasigroups. - Kishinev: Shtintsa, 1972. - 225 p.
57. Sokhatsky F. M. On the associativity of multiplace operations // Quasigroups Relat. Syst. - 1997. - Vol. 4. - P. 51–66.
58. Michalski J. Covering k-groups of n-groups // Arch. Math. (Brno). - 1981. - Vol. 17. - № 4. - P. 207–226.
59. Pop M. S., Pop A. On some relations on n-monoids // Carpathian J. Math. - 2004. - Vol. 20. - № 1. - P. 87–94.
60. Čupona G., Trpenovski B. Finitary associative operations with neutral elements // Bull. Soc. Math. Phys. Macedoine. - 1961. - Vol. 12. - P. 15–24.
61. Evans T. Abstract mean values // Duke Math J. - 1963. - Vol. 30. - P. 331–347.
62. Głazek K., Gleichgewicht B. Abelian n-groups // Colloq. Math. Soc. J. Bolyai. Universal Algebra (Esztergom, 1977). - Amsterdam. North-Holland, 1982. - P. 321–329.
63. Stojaković Z., Dudek W. A. On σ-permutable n-groups // Publ. Inst. Math., Nouv. Sér. - 1986. - Vol. 40(54). - P. 49–55.
64. Rusakov S. A. Some Applications of n-ary Group Theory. - Minsk: Belaruskaya navuka, 1998. - 180 p.
65. Celakoski N. On some axiom systems for n-groups // Mat. Bilt. - 1977. - Vol. 1. - P. 5–14.
66. Gal'mak A. M. n-ary Groups, Part 1. - Gomel: Gomel University, 2003. - 195 p.
67. Ŭsan J. n-groups in the light of the neutral operations // Math. Moravica. - 2003. - Vol. Special vol.
68. Sokolov E. I. On the theorem of Gluskin-Hosszú on Dörnte groups // Mat. Issled. - 1976. - Vol. 39. - P. 187–189.

69. Yurevych O. V. Criteria for invertibility of elements in associates // Ukr. Math. J. - 2001. - Vol. 53. - № 11. - P. 1895–1905.
70. Timm J. Verbandstheoretische Behandlung n-stelliger Gruppen // Abh. Math. Semin. Univ. Hamb. - 1972. - Vol. 37. - P. 218–224.
71. Dudek W. A. Remarks on n-groups // Demonstratio Math. - 1980. - Vol. 13. - P. 165–181.
72. Dudek W. A. Autodistributive n-groups // Annales Sci. Math. Polonae, Commentationes Math. - 1993. - Vol. 23. - P. 1–11.
73. Heine E. Handbuch der Kugelfunktionan. - Berlin: Reimer, 1878.
74. Kac V., Cheung P. Quantum calculus. - New York: Springer, 2002. - 112 p.
75. Petrescu A. On the homotopy of universal algebras. I. // Rev. Roum. Math. Pures Appl. - 1977. - Vol. 22. - P. 541–551.
76. Halaš R. A note on homotopy in universal algebra // Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math. - 1994. - Vol. 33. - № 114. - P. 39–42.
77. Głazek K., Gleichgewicht B. Bibliography of n-groups (polyadic groups) and some group like n-ary systems // Proceedings of the Symposium on n-ary structures. - Skopje. Macedonian Academy of Sciences and Arts, 1982. - P. 253–289.
78. Fujiwara T. On mappings between algebraic systems // Osaka Math. J. - 1959. - Vol. 11. - P. 153–172.
79. Vidal J. C., Tur J. S. A 2-categorial generalization of the concept of institution // Stud. Log. - 2010. - Vol. 95. - № 3. - P. 301–344.
80. Novotný M. Homomorphisms of algebras // Czech. Math. J. - 2002. - Vol. 52. - № 2. - P. 345–364.
81. Novotný M. Mono-unary algebras in the work of Czechoslovak mathematicians // Arch. Math., Brno. - 1990. - Vol. 26. - № 2-3. - P. 155–164.
82. Gal'mak A. M. Generalized morphisms of algebraic systems // Vopr. Algebry. - 1998. - Vol. 12. - P. 36–46.
83. Gal'mak A. M. Generalized morphisms of Abelian m-ary groups // Discuss. Math. - 2001. - Vol. 21. - № 1. - P. 47–55.
84. Gal'mak A. M. n-ary Groups, Part 2. - Minsk: Belarus State University, 2007. - 324 p.
85. Goetz A. On weak isomorphisms and weak homomorphisms of abstract algebras // Colloq. Math. - 1966. - Vol. 14. - P. 163–167.
86. Marczewski E. Independence in abstract algebras. Results and problems // Colloq. Math. - 1966. - Vol. 14. - P. 169–188.
87. Głazek K., Michalski J. On weak homomorphisms of general non-indexed algebras // Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. - 1974. - Vol. 22. - P. 651–656.
88. Głazek K. Weak homomorphisms of general algebras and related topics // Math. Semin. Notes, Kobe Univ. - 1980. - Vol. 8. - P. 1–36.
89. Traczyk T. Weak isomorphisms of Boolean and Post algebras // Colloq. Math. - 1965. - Vol. 13. - P. 159–164.
90. Denecke K., Wismath S. L. Universal Algebra and Coalgebra. - Singapore: World Scientific, 2009.
91. Denecke K., Saengsura K. State-based systems and (F_1, F_2)-coalgebras // East-West J. Math.,. - 2008. - Vol. Spec. Vol. - P. 1–31.
92. Chung K. O., Smith J. D. H. Weak homomorphisms and graph coalgebras // Arab. J. Sci. Eng., Sect. C, Theme Issues. - 2008. - Vol. 33. - № 2. - P. 107–121.
93. Kolibiar M. Weak homomorphisms in some classes of algebras // Stud. Sci. Math. Hung. - 1984. - Vol. 19. - P. 413–420.
94. Głazek K., Michalski J. Weak homomorphisms of general algebras // Commentat. Math. - 1977. - Vol. 19 (1976). - P. 211–228.
95. Czákány B. On the equivalence of certain classes of algebraic systems // Acta Sci. Math. (Szeged). - 1962. - Vol. 23. - P. 46–57.
96. Mal'tcev A. I. Free topological algebras // Izv. Akad. Nauk SSSR. Ser. Mat. - 1957. - Vol. 21. - P. 171—-198.
97. Mal'tsev A. I. The structural characteristic of some classes of algebras // Dokl. Akad. Nauk SSSR. - 1958. - Vol. 120. - P. 29–32.
98. Ellerman D. A theory of adjoint functors – with some thoughts about their philosophical significance // What is category theory? - Monza (Milan). Polimetrica, 2006. - P. 127–183.
99. Ellerman D. Adjoints and emergence: applications of a new theory of adjoint functors // Axiomathes. - 2007. - Vol. 17. - № 1. - P. 19–39.
100. csi B. P. On Morita contexts in bicategories // Applied Categorical Structures. - 2011. - P. 1–18.
101. Chronowski A. A ternary semigroup of mappings // Demonstr. Math. - 1994. - Vol. 27. - № 3-4. - P. 781–791.
102. Chronowski A. On ternary semigroups of homomorphisms of ordered sets // Arch. Math. (Brno). - 1994. - Vol. 30. - № 2. - P. 85–95.
103. Chronowski A., Novotný M. Ternary semigroups of morphisms of objects in categories // Archivum Math. (Brno). - 1995. - Vol. 31. - P. 147–153.
104. Gal'mak A. M. Polyadic associative operations on Cartesian powers // Proc. of the Natl. Academy of Sciences of Belarus, Ser. Phys.-Math. Sci. - 2008. - Vol. 3. - P. 28–34.
105. Gal'mak A. M. Polyadic analogs of the Cayley and Birkhoff theorems // Russ. Math. - 2001. - Vol. 45. - № 2. - P. 10–15.
106. Gleichgewicht B., Wanke-Jakubowska M. B., Wanke-Jerie M. E. On representations of cyclic n-groups // Demonstr. Math. - 1983. - Vol. 16. - P. 357–365.
107. Wanke-Jakubowska M. B., Wanke-Jerie M. E. On representations of n-groups // Annales Sci. Math. Polonae, Commentationes Math. - 1984. - Vol. 24. - P. 335–341.
108. Berezin F. A. Introduction to Superanalysis. - Dordrecht: Reidel, 1987. - 421 p.
109. Kirillov A. A. Elements of the Theory of Representations. - Berlin: Springer-Verlag, 1976.
110. Husemöller D., Joachim M., Jurčo B., Schottenloher M. G-spaces, G-bundles, and G-vector bundles // Basic Bundle Theory and K-Cohomology Invariants. - Berlin-Heidelberg. Springer, 2008. - P. 149-161.
111. Mal'tcev A. I. On general theory of algebraic systems // Mat. Sb. - 1954. - Vol. 35. - № 1. - P. 3–20.
112. Gal'mak A. M. Translations of n-ary groups. // Dokl. Akad. Nauk BSSR. - 1986. - Vol. 30. - P. 677–680.
113. Zeković B., Artamonov V. A. A connection between some properties of n-group rings and group rings // Math. Montisnigri. - 1999. - Vol. 15. - P. 151–158.
114. Zeković B., Artamonov V. A. On two problems for n-group rings // Math. Montisnigri. - 2002. - Vol. 15. - P. 79–85.
115. Carlsson R. N-ary algebras // Nagoya Math. J. - 1980. - Vol. 78. - № 1. - P. 45–56.
116. Bremner M., Hentzel I. Identities for generalized lie and jordan products on totally associative triple systems // J. Algebra. - 2000. - Vol. 231. - № 1. - P. 387–405.
117. Madore J. Introduction to Noncommutative Geometry and its Applications. - Cambridge: Cambridge University Press, 1995.
118. Kogorodski L. I., Soibelman Y. S. Algebras of Functions on Quantum Groups. - Providence: AMS, 1998.
119. Chari V., Pressley A. A Guide to Quantum Groups. - Cambridge: Cambridge University Press, 1996.
120. Majid S. Foundations of Quantum Group Theory. - Cambridge: Cambridge University Press, 1995.
121. Drinfeld V. G. Quantum groups // Proceedings of the ICM, Berkeley. - Phode Island. AMS, 1987. - P. 798–820.

Steven Duplij (Stepan Anatolievich Douplii) is a theoretical physicist, a Lead Staff Researcher of the Theory Group, Nuclear Physics Laboratory at V.N. Karazin Kharkov National University, and Doctor of Physical and Mathematical Sciences. He has more than hundred scientific publications, several monographs. The main scientific directions are supersymmetry and supermanifolds, quantum groups and their actions, singular theories and constrained systems, nonlinear and conformal electrodynamics, multigravity and Pauli-Fierz models, ternary and polyadic structures, von Neumann regularity and semigroups, DNA theory and genetic code. Editor-compiler of "Concise Encylopedia of Supersymmetry"(Springer), he has received grants from the Fulbright and Humboldt Foundations.