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Bachelor thesis

Continued fractions with restricted digits and their Hausdorff dimension

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Abstract

In 2001 Jenkinson and Pollicott developed an algorithm to compute the Hausdorff dimension of fractal sets constructed by the Gauss map. This thesis presents the theory required to establish correctness of Jenkinson-Pollicott's algorithm. Jenkinson and Pollicott proved that the convergence rate of their algorithm is super-exponential. This thesis improves this rate to

$$\mathcal{O}\left(\sqrt[2|\mathcal{A}|]{\frac{4+2p}{5+2p}}^{n^2}\right) \tag{0.1}$$

where $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ is a finite set with $|\mathcal{A}| \geq 2$, $p = \max \mathcal{A}$ and $n + 1 \in \mathbb{N}_{\geq 1}$ is the number of terms used in the Taylor approximation of equation (3.3). The thesis concludes with a practical use of the algorithm, namely the attempt to prove Zaremba's conjecture done by Bourgain and Kontorovich.

Contents

Chapter 1

Introduction

In 1972 Zaremba [Zar72] conjectured the following statement:

Zaremba's conjecture. *Let $A \in \mathbb{N}_{\geq 1}$ be an integer and let $\mathcal{A} = \{1, \ldots, A\}$ be a finite alphabet. Let*

$$\mathscr{R}_A = \left\{ x \in [0,1] \;\middle|\; \exists \{a_i\}_{i=1}^{k} \subset \mathcal{A} : x = \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}} \right\} \tag{1.1}$$

be the set of finite continued fractions with coefficients in $\mathcal{A}$. Let

$$\mathscr{D}_A = \left\{ d \in \mathbb{N}_{\geq 1} \;\middle|\; \exists b \in \mathbb{N} : \frac{b}{d} \in \mathscr{R}_A \wedge \gcd(b,d) = 1 \right\} \tag{1.2}$$

be the set of the denominators of the elements in $\mathscr{R}_A$. Then there is a $Z \in \mathbb{N}$ such that $\mathscr{D}_Z = \mathbb{N}_{\geq 1}$.

This conjecture is the motivation for this thesis. Many have tried to prove it, but no one has yet succeeded. The closest result to proving Zaremba's conjecture is by Bourgain and Kontorovich [BK14] and Huang [Hua15] where in the last article Huang proved

$$\lim_{N \to \infty} \frac{1}{N} \left| \mathscr{D}_5 \cap [1, N] \right| = 1. \tag{1.3}$$

To develop the proof of this conjecture further we need efficient tools to compute the Hausdorff dimension of certain fractal sets.

In this thesis we let $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ be a finite set with $|\mathcal{A}| \geq 2$ and we consider the set

$$E_{\mathcal{A}} = \left\{ x \in [0,1] \;\middle|\; \exists \{a_i\}_{i=1}^{\infty} \subset \mathcal{A} : x = \cfrac{1}{a_1 + \cfrac{1}{\ddots}} \right\}. \tag{1.4}$$

The set $E_{\mathcal{A}}$ is a fractal set with a certain Hausdorff dimension. Bourgain and Kontorovich [BK14] used the Hausdorff dimension of $E_{\{1,2\}}$ to prove their main result, here Theorem 4.1.2 on page 28. The problem of computing $\dim_H\left(E_{\{1,2\}}\right)$ traces back to Jarník [Jar29] who proved $\dim_H\left(E_{\{1,2\}}\right) > \frac{1}{4}$. Good [Goo41] improved this estimation to $0.5194 \le \dim_H\left(E_{\{1,2\}}\right) \le 0.5433$. Bumby [Bum82] proved 41 years later $0.5312 \le \dim_H\left(E_{\{1,2\}}\right) \le 0.5314$. Hensley [Hen89] improved this bound further in 1989 and in 1996 he derived a polynomial time algorithm for computing $\dim_H\left(E_{\{1,2\}}\right)$ [Hen96]. With this algorithm the estimate was improved to $\dim_H\left(E_{\{1,2\}}\right) \approx 0.53128\ldots$ which is accurate up to 19 decimal places. Jenkinson and Pollicott [JP01] constructed an algorithm that computes $\dim_H\left(E_{\{1,2\}}\right)$ with super-exponential convergence rate and in [Jen04] Jenkinson computes $\dim_H\left(E_{\{1,2\}}\right)$ accurately up to 54 decimal places.

Bourgain and Kontorovich [BK14] used Jenkinson-Pollicott's algorithm derived in [JP01, Jen04], here algorithm 1 on page 20, to compute $\dim_H\left(E_{\mathcal{A}}\right)$ for several finite non-empty $\mathcal{A} \subset \mathbb{N}_{\ge 1}$. This algorithm draws heavily on the work of Ruelle [Rue76] and Bowen [Bow79] who first used thermodynamical formalisms to compute the Hausdorff dimension of certain sets. In [JP01] they proved that their algorithm is correct for $\mathcal{A} = \{1,2\}$. In [Jen04] Jenkinson generalizes his algorithm for all finite non-empty $\mathcal{A}$, but he does not prove the correctness of his generalized algorithm. Moreover, if n is the number of terms in the Taylor approximation of equation (3.3) around $z = 0$, then in [JP01] they proved that the convergence rate of their algorithm for $\dim_H\left(E_{\{1,2\}}\right)$ is

$$\mathcal{O}\left(\sqrt[4]{\frac{8}{9}}^{n^2}\right), \tag{1.5}$$

but in [Jen04] Jenkinson states that the convergence rate when computing $\dim_H\left(E_{\mathcal{A}}\right)$ with finite non-empty $\mathcal{A} \subset \mathbb{N}_{\ge 1}$ is $\mathcal{O}\left(\theta^{n^2}\right)$ with $\theta \in (0,1)$ unspecified.

This thesis considers the correctness of Jenkinson-Pollicott's algorithm and its convergence rate. Theorem 3.2.2 proves that Jenkinson-Pollicott's algorithm is correct for all finite $\mathcal{A} \subset \mathbb{N}_{\ge 1}$ with $|\mathcal{A}| \ge 2$. For $|\mathcal{A}| = 1$ the set $E_{\mathcal{A}}$ consists of a single point which has zero Hausdorff dimension [Fal90, JP01]. Thus $\dim_H\left(E_{\mathcal{A}}\right)$ can now be computed for all finite non-empty $\mathcal{A} \subset \mathbb{N}_{\ge 1}$. Theorem 3.2.8 proves that the convergence rate of Jenkinson-Pollicott's algorithm is

$$\mathcal{O}\left(\sqrt[2|\mathcal{A}|]{\frac{4+2p}{5+2p}}^{n^2}\right) \tag{1.6}$$

where $\mathcal{A} \subset \mathbb{N}_{\ge 1}$ is finite, $|\mathcal{A}| \ge 2$ and $p = \max \mathcal{A}$ and $n+1$ is the number of terms of the Taylor approximation of equation (3.28) used in algorithm 1, specifying the earlier mentioned θ.

In chapter 2 we will develop theory on the Hausdorff dimension of general sets and $E_{\mathcal{A}}$ following [Bed91, Bow79, Fal90]. In chapter 3 we first consider the Ruelle operator [Rue76] in the context of continued fractions [JP01]. Then we use the Ruelle operator to prove the main result of this thesis, namely Theorem 3.2.2 and 3.2.8. Chapter 4 considers Bourgain's and Kontorovich's article [BK14] and how they used $\dim_H\left(E_{\{1,2\}}\right)$ in the proof of Theorem 4.1.1 from where their main result Theorem 4.1.2 follows.

Chapter 2

Preliminaries

This chapter concerns all the necessary information and background knowledge needed for the thesis. The thesis is based around the Gauss map with restricted digits, which is the first topic we will discuss. Then the main definition of the thesis will be given, namely the definition of Hausdorff dimension. Lastly some techniques for computing Hausdorff dimension and the notion of pressure will be introduced.

2.1 Gauss map

The whole thesis is centered around the following transformation:

Definition 2.1.1 (Gauss map)**.** Let $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ be finite and non-empty. The *Gauss map with restricted digits* is given by

$$T_{\mathcal{A}}\colon \bigcup_{a \in \mathcal{A}} \left[\frac{1}{a+1}, \frac{1}{a}\right] \to [0,1], \qquad x \mapsto \frac{1}{x} \mod 1. \tag{2.1}$$

Let $a \in \mathcal{A}$ and define $T_a\colon [0,1] \to \left[\frac{1}{a+1}, \frac{1}{a}\right]$ by $x \mapsto \frac{1}{x+a}$ as the *a-th inverse branch* of the Gauss map. The set $\mathcal{A}$ is called an *alphabet*.

In the rest of the thesis $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ will be a finite and non-empty alphabet. In figure 2.1 an example of the Gauss-map with digits in $\mathcal{A} = \{1, 2, 4, 6\}$ is given to clarify the definition.

The real Gauss map, made by Gauss, is $x \mapsto \frac{1}{x} \mod 1$ with $x \in [0,1]$. In Jenkinson's article [Jen04] the map of Definition 2.1.1 is called Gauss map and in this thesis we continue to do so. The Gauss map with $\mathcal{A} = \{1, 2\}$ is researched in many articles such as [Jar29, Goo41, Bum82, Hen89, Hen96, JP01, Jen04]. This thesis draws mainly on the work from [JP01, Jen04]. It is not known in which article Gauss himself defined the Gauss map.

Consider the set $E_{\mathcal{A}} := \bigcap_{n=0}^{\infty} T_{\mathcal{A}}^{-n}([0,1]) \subset [0,1]$. This set is called the *repeller* of map $T_{\mathcal{A}}$. In [JP01, Jen04] it is stated that the set $E_{\mathcal{A}}$ is the set of

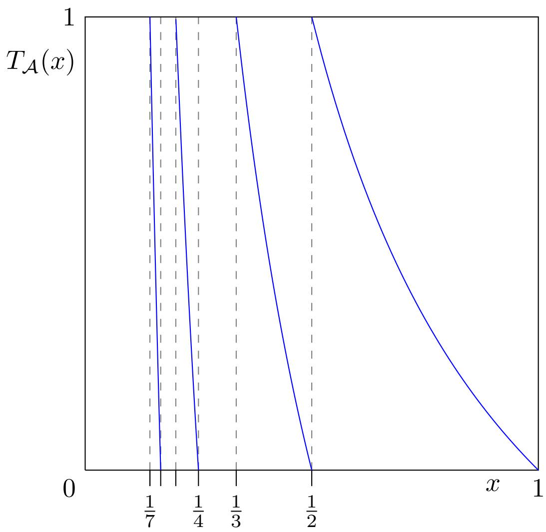

Figure 2.1: Example of the Gauss map for $\mathcal{A} = \{1, 2, 4, 6\}$.

all continued fractions with coefficients in $\mathcal{A}$:

$$E_{\mathcal{A}} = \left\{ x \in [0,1] \;\middle|\; \exists \{a_i\}_{i=1}^{\infty} \in \mathcal{A} : x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} \right\}. \tag{2.2}$$

This can be verified. Let $\{a_i\}_{i=1}^{\infty} \in \mathcal{A}$ be an arbitrary sequence and let

$$x = [a_1, a_2, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} \in [0,1]. \tag{2.3}$$

Then $T_{\mathcal{A}}(x) = [a_1, a_2, \ldots]^{-1} \mod 1 = a_1 + [a_2, a_3, \ldots] \mod 1 = [a_2, a_3, \ldots]$ holds, thus $T_{\mathcal{A}}(x)$ is also an element of $E_{\mathcal{A}}$. Writing out $E_{\mathcal{A}} = \bigcap_{n=0}^{\infty} T_{\mathcal{A}}^{-n}([0,1])$ gives $E_{\mathcal{A}}$ from (2.2). Note that $E_{\mathcal{A}}$ can be generated from the maps T_a for all $a \in \mathcal{A}$ [JP01].

2.2 Hausdorff dimension

2.2.1 Definition

This thesis discusses the computation of the Hausdorff dimension of $E_{\mathcal{A}}$ for many alphabets $\mathcal{A}$. Therefore we first need to define Hausdorff dimension, which will be done in Definition 2.2.4. The definitions and notation used are mostly from the book [Fal90] written by Falconer.

Definition 2.2.1 (δ-cover)**.** Let $E \subset \mathbb{R}^n$ be a Borel-measurable set. Denote the *diameter* of E as $\text{diam}(E) = \sup_{x,y \in E} |x - y|$. Let $\{U_i\}_{i=1}^{\infty} \subset \mathcal{P}(\mathbb{R}^n)$ be a countable family of open Borel-measurable sets. Let $\delta \in \mathbb{R}_{>0}$, then $\{U_i\}_{i=1}^{\infty}$ is a *δ-cover of E* if $E \subset \bigcup_{i=1}^{\infty} U_i$ and $\text{diam}(U_i) \in (0, \delta)$ for all $i \in \mathbb{N}_{\geq 1}$ hold.

A special measure called s-dimensional Hausdorff measure is needed to define the Hausdorff dimension. The s-dimensional Hausdorff measure measures the size of δ-covers of dimension s needed to cover a set E. This will be made precise in the following definition.

Definition 2.2.2 (Hausdorff measure)**.** Let $s \in \mathbb{R}_{\geq 0}$ and $\delta \in \mathbb{R}_{>0}$. Let $E \subset \mathbb{R}^n$ be a Borel-measurable set. Define the map $\mathcal{H}_\delta^s \colon \mathcal{B}(\mathbb{R}^n) \to [0, \infty]$ by

$$\mathcal{H}_\delta^s(E) = \inf \left\{ \sum_{i=1}^{\infty} \text{diam}\left(U_i\right)^s \,\middle|\, \{U_i\}_{i=1}^{\infty} \subset \mathcal{B}\left(\mathbb{R}^n\right) \text{ is a } \delta\text{-cover of } E \right\}. \tag{2.4}$$

The *s-dimensional Hausdorff measure* $\mathcal{H}^s$ is defined as

$$\mathcal{H}^s := \lim_{\delta \to 0} \mathcal{H}_\delta^s. \tag{2.5}$$

It is proven in [Fal90] that $\mathcal{H}^s$ really is a measure for all $s \in \mathbb{R}_{\geq 0}$. The following proposition proven in [Fal90] is of utmost importance for the definition of Hausdorff dimension.

Proposition 2.2.3. *Let $s \in \mathbb{R}_{\geq 0}$ with $t > s$ and $\delta \in \mathbb{R}_{>0}$. Let $E \subset \mathbb{R}_{\geq 0}^n$ be Borel-measurable. If $\mathcal{H}^s(E)$ is finite, then $\mathcal{H}^t(E)$ is zero for all $t > s$.*

The Hausdorff dimension of a Borel-measurable set $E \subset \mathbb{R}^n$ is the $s \in \mathbb{R}_{\geq 0}$ where the map $t \mapsto \mathcal{H}^t(E)$ jumps from ∞ to 0.

Definition 2.2.4 (Hausdorff dimension)**.** Let $E \subset \mathbb{R}^n$ be Borel-measurable. The *Hausdorff dimension of E* is defined as

$$\dim_H(E) := \inf \left\{ s \in \mathbb{R}_{\geq 0} \,|\, \mathcal{H}^s(E) = 0 \right\}. \tag{2.6}$$

Most of the times after such a definition an example would be given. However, computing the Hausdorff dimension directly by definition is extremely hard for even the simplest fractals. Therefore we first discuss some computational techniques and then in Examples 2.2.11 on page 9 and 2.2.15 on page 12 the Hausdorff dimension of the middle-third Cantor set will be efficiently computed.

Suppose we know the Hausdorff dimension of some fractals in $\mathbb{R}^n$ and we want to compute the Hausdorff dimension of their union, then we can use the following property proven by [Fal90].

Property 2.2.5. *Let $\{E_i\}_{i=1}^{\infty} \subset \mathbb{R}^n$ be a family of Borel-measurable sets. Then*

$$\dim_H\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_{i \in \mathbb{N}} \dim_H(E_i). \tag{2.7}$$

2.2.2 Computational techniques for Hausdorff dimension

There are a few computational techniques we can use for the Hausdorff dimension. In this subsection the mass-distribution principle, box-counting dimension and pressure will be introduced which provide ease in the computation of the Hausdorff dimension.

Mass-distribution principle

Finding a lower bound requires proving that every δ-cover for every arbitrary small $\delta \in \mathbb{R}_{>0}$ rises above the required infimum in the definition of Hausdorff dimension. The mass-distribution principle, which is proven in [Fal90], enables us to compute a lower bound for the Hausdorff dimension with less effort.

Proposition 2.2.6 (Mass-distribution principle)**.** *Let $E \subset \mathbb{R}^n$ be a Borel-measurable set. Let μ be a probability measure on E and suppose that for some $s \in \mathbb{R}_{\geq 0}$ there exist $c, \delta \in \mathbb{R}_{>0}$ such that $\mu(U) \leq c \cdot \operatorname{diam}(U)^s$ holds for all Borel-measurable $U \subset E$ with $\operatorname{diam}(U) \leq \delta$. Then $\mathcal{H}^s(E) \geq \frac{1}{c}$ and $\dim_H E \geq s$.*

The mass-distribution principle can in our case be specified to the pointwise Hausdorff dimension of a probability measure. In section 2 of article [BW06] the following result is proven:

Proposition 2.2.7 (Pointwise Hausdorff dimension)**.** *Let $E \subset \mathbb{R}^n$ be a Borel-measurable set and let μ be a probability measure on E. The* lower and upper pointwise Hausdorff dimension of μ in x *are respectively for every $x \in E$ defined as*

$$\underline{d}_\mu(x) = \liminf_{s \to 0} \frac{\log \mu(B(x,s))}{\log s}, \qquad \overline{d}_\mu(x) = \limsup_{s \to 0} \frac{\log \mu(B(x,s))}{\log s} \tag{2.8}$$

where $B(x,s)$ is a ball in E with center x and radius s.
If $\underline{d}_\mu(x) = \overline{d}_\mu(x)$ holds for an $x \in E$, then the pointwise Hausdorff dimension in x *is defined as $d_\mu(x) = \underline{d}_\mu(x)$. If there is an $s \in \mathbb{R}_{\geq 0}$ such that $d_\mu(x) = s$ holds for μ-almost every $x \in E$, then $\dim_H(E) = s$. If $\underline{d}_\mu(x) \geq s$ holds for an $s \in \mathbb{R}_{\geq 0}$ and μ-almost every $x \in E$, then $\dim_H(E) \geq s$.*

Proposition 2.2.7 is useful in computing a lower bound for the Hausdorff dimension of fractals. This will be demonstrated in Example 2.2.11.

Box-counting dimension

Finding an upper bound for the Hausdorff dimension by definition can be challenging, as the definition is not easy to work with. Therefore another dimension called box-counting dimension that is easier to compute will be introduced. The box-counting dimension is an upper bound for the Hausdorff dimension.

To find an upper bound for the Hausdorff dimension one can cover the fractal with boxes of a specific length. Then one can count the number of boxes and investigate the ratio of the number of boxes needed and the length of a box. This gives rise to the box-counting dimension, which is given by Falconer in [Fal90].

Definition 2.2.8 (Box-counting dimension)**.** Let $E \subset \mathbb{R}^n$ be Borel-measurable. Let $N_\delta(E)$ be any of the following:

1. The smallest number of closed balls of radius δ that cover E;
2. The smallest number of cubes of side δ that cover E;
3. The smallest number of sets of diameter at most δ that cover E;
4. The largest number of disjoint balls of radius δ with center in E.

The *lower box-counting dimension of* E is given by

$$\underline{\dim}_B(E) = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}. \tag{2.9}$$

The *upper box-counting dimension of* E is given by

$$\overline{\dim}_B(E) = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}. \tag{2.10}$$

When the lower- and upper box-counting dimension of E exists, the *box-counting dimension* is given by $\dim_B(E) = \underline{\dim}_B(E) = \overline{\dim}_B(E)$.

The box-counting dimension is not the same as the Hausdorff dimension. This can be proven by the following proposition from [Fal90]:

Proposition 2.2.9. *Let $E \subset \mathbb{R}^n$ be Borel-measurable. Denote $\overline{E}$ as the closure of E. Then $\underline{\dim}_B(\overline{E}) = \underline{\dim}_B(E)$ and $\overline{\dim}_B(\overline{E}) = \overline{\dim}_B(E)$ hold.*

Consider $\mathbb{Q}$ and its Hausdorff and box-counting dimension. The set $\mathbb{Q}$ is countable and $\dim_H(\{x\})$ is zero for every $x \in \mathbb{Q}$. Therefore according to Property 2.2.5 the Hausdorff dimension of $\mathbb{Q}$ equals

$$\dim_H(\mathbb{Q}) = \dim_H\left(\bigcup_{x \in \mathbb{Q}} \{x\}\right) = \sup_{x \in \mathbb{Q}} \dim_H(\{x\}) = 0. \tag{2.11}$$

Note that $\overline{\mathbb{Q}} = \mathbb{R}$ holds and it can easily be proven that $\dim_B(\mathbb{R}) = 1$. Thus $\dim_B(\mathbb{Q}) = \dim_B(\mathbb{R}) = 1$ holds. Thus the box-counting dimension of $\mathbb{Q}$ is larger than the Hausdorff dimension of $\mathbb{Q}$. This will be made precise by the next proposition from [Fal90].

Proposition 2.2.10. *Let $E \subset \mathbb{R}^n$ be Lebesgue-measurable. Suppose that for all $k \in \mathbb{N}_{\geq 1}$ the set E can be covered by n_k sets of diameter at most δ_k with $\lim_{k \to \infty} \delta_k = 0$. Then*

$$\dim_H(E) \leq \underline{\dim}_B(E) \leq \liminf_{k \to \infty} \frac{\log n_{\delta_k}(E)}{-\log \delta_k}. \tag{2.12}$$

Now follows an example where the Hausdorff dimension of the middle-third Cantor set will be computated. This example demonstrates all the theory that is developed in section 2.2.

Example 2.2.11 (Middle-third Cantor set)**.** Let $\{E_n\}_{n=0}^{\infty} \subset [0,1]$ be a family of sets where $E_0 = [0,1]$ and E_{n+1} can be obtained from E_n for all $n \in \mathbb{N}$ by splitting every interval in E_n in 3 equal-sized intervals and removing the middle interval. Define the middle-third Cantor set as $F := \bigcap_{n=0}^{\infty} E_n$. An illustration of this process can be found in figure 2.2.

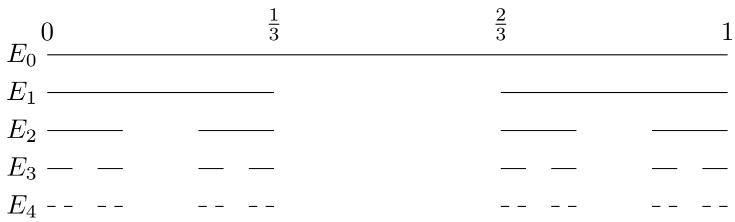

Figure 2.2: Construction of the middle-third Cantor set up to $n = 4$.

First we will compute an upper bound and then a lower bound for $\dim_H(F)$. Let $n \in \mathbb{N}$ be arbitrary and consider E_n. Then there are exactly 2^n intervals of size 3^{-n} needed to cover E_n where the intersection is empty. Now we can use Proposition 2.2.10 to provide an upper bound for $\dim_H(F)$. Let $\{\delta_k\}_{k=0}^{\infty} \subset \mathbb{R}_{>0}$ be a sequence with $\delta_k = 3^{-k}$ for all $k \in \mathbb{N}$. Then the sequence $\{n_k\}_{n=0}^{\infty} \subset \mathbb{N}$ must have $n_k = 2^k$ for all $k \in \mathbb{N}$ according to Proposition 2.2.10. Since the intervals needed to cover E_n also cover F for all $n \in \mathbb{N}$, Proposition 2.2.10 gives

$$\dim_H(F) \leq \liminf_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k} = \liminf_{k\to\infty} \frac{\log 2^k}{-\log 3^{-k}} = \log_3 2. \tag{2.13}$$

For the lower bound Proposition 2.2.7 will be used. Let $n \in \mathbb{N}$ be arbitrary and consider E_n. Let $x \in E_n \setminus \partial E_n$ be arbitrary. Let $s' \in \mathbb{R}_{>0}$ be such that $B(x, s') \subset E_n$. Then $B(x, s) \subset E_n$ holds for all $s \in [0, s']$. Let μ_n be the uniform Borel-probability measure on E_n, thus $\mu_n(X) = \frac{\lambda^1(X)}{\lambda^1(E_n)}$ for all Borel-measurable $X \subseteq E_n$ where λ^1 is the 1-dimensional Lebesgue measure. Now Proposition 2.2.7 can be applied. Note that $s' \leq \frac{1}{2}3^{-n}$ must hold. Thus

$$\begin{aligned}
\underline{d}_{\mu_n}(x) &= \liminf_{s\to 0} \frac{\log \mu_n\left(B\left(x, s\right)\right)}{\log s} = \liminf_{s\to 0} \frac{\log \mu_n\left(\left(x - s, x + s\right)\right)}{\log s} && (2.14)\\
&= \liminf_{s\to 0} \frac{\log \frac{2s}{2^n 3^{-n}}}{\log s} = \liminf_{s\to 0} \left(\frac{\log s + \log 2}{\log s} + \frac{n\log 3 - n\log 2}{\log s}\right) && (2.15)\\
&\geq 1 + \liminf_{s\to 0} \frac{n\log 3 - n\log 2}{\log \frac{1}{2}3^{-n}} = 1 - \frac{n\log 3 - n\log 2}{n\log 3 + \log 2}. && (2.16)
\end{aligned}$$

Let now $x \in F$. Taking $n \to \infty$ gives

$$\begin{aligned}
\lim_{n\to\infty} \underline{d}_{\mu_n}(x) &\geq 1 - \lim_{n\to\infty} \frac{n\log 3 - n\log 2}{n\log 3 + \log 2} && (2.17)\\
&= 1 - \frac{\log 3 - \log 2}{\log 3} = 1 - 1 + \log_3 2. && (2.18)
\end{aligned}$$

Therefore $\underline{d}_{\lim\limits_{n\to\infty} \mu_n}(x) \geq \log_3 2$ holds for $\lim\limits_{n\to\infty} \mu_n$-almost every $x \in F$. Thus $\dim_H(F) \geq \log_3 2$ holds.

Now we have proven that $\dim_H(F)$ is bounded from below and above by $\log_3 2$. Thus the Hausdorff dimension of the middle-third Cantor set is $\log_3 2$.

2.2.3 Pressure

The middle-third Cantor set is constructed with equally sized disjoint intervals which enables an easy covering. This is not the case for $E_{\mathcal{A}}$, thus we need more techniques to compute the Hausdorff dimension of $E_{\mathcal{A}}$ in addition to the ones developed in section 2.2.

Bowen [Bow79] first introduced pressure to analyze the dimension of quasi-circles. Mathematicians as Ruelle [Rue76, Rue78], McCluskey, Manning [MM83] and Bedford [Bed91] applied the approach of pressure to other cases. Here we will use the approach in Bedford's article [Bed91] as described in [JP01] by Jenkinson and Pollicott. Using the notion of pressure Bowen proved Theorem 2.2.14 also known as Bowen's formula [Bow79]. Jenkinson and Pollicott then use this formula in [JP01, Jen04] to find an efficient algorithm to approximate the Hausdorff dimension of $E_{\mathcal{A}}$. Therefore, the rest of the thesis is centered around Theorem 2.2.14.

Definition 2.2.12 (Pressure)**.** Let $f\colon E_{\mathcal{A}} \to \mathbb{R}$ be a continuous function and let $T_{\mathcal{A}}$ be the restricted Gauss map from Definition 2.1.1. The *pressure of f with respect to $T_{\mathcal{A}}$* is given by

$$P(f) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n_{\mathcal{A}}(x)=x} \exp\left(\sum_{k=0}^{n-1} f\left(T^k_{\mathcal{A}}(x)\right)\right). \tag{2.19}$$

Two properties of pressure can be derived from the definition.

Property 2.2.13. *Let $f, g\colon E_{\mathcal{A}} \to \mathbb{R}$ be continuous functions. Pressure has the following two properties:*

1. *If $f < g$ holds, then $P(f) \leq P(g)$ holds.*
2. *The map $f \mapsto P(f)$ is convex with respect to f.*

The connection between pressure and Hausdorff dimension is given in the following theorem.

Theorem 2.2.14 (Bowen's formula)**.** *Let $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ be finite and non-empty. Consider the function $s \mapsto P\left(-s \log |T'_{\mathcal{A}}|\right)$ with $s \in [0,1]$. Then the unique solution s' to*

$$P\left(-s\log|T'_{\mathcal{A}}|\right) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{T^n_{\mathcal{A}}(x)=x \\ x\in E_{\mathcal{A}}}} \prod_{k=0}^{n-1} \left(T^k_{\mathcal{A}}(x)\right)^{2s} = 0 \tag{2.20}$$

is equal to $s' = \dim_H(E_{\mathcal{A}})$.

Proof. The proof is given in [Bed91]. Even a scheme of the proof is out of the scope of this thesis as it involves entropy. The unique zero can be proven in the following way. Writing out $P\left(-s\log|T'_{\mathcal{A}}|\right)$ and using Property 2.2.13 shows that $s \mapsto P\left(-s\log|T'_{\mathcal{A}}|\right)$ is a continuous map and strictly decreasing in s. Using the theory of entropy, one can prove that $P(0) > 0$ and $P\left(\log|T'_{\mathcal{A}}|^{-1}\right) < 0$ holds, thus the Intermediate Value Theorem gives us a unique zero. □

To show the power of Theorem 2.2.14 we will compute the Hausdorff dimension of the middle-third Cantor set once again in Example 2.2.15.

Example 2.2.15 (Middle-third Cantor set)**.** Let $I_1, I_2 \subset [0,1]$ be arbitrary closed disjoint intervals. Let $S\colon I_1 \cup I_2 \to [0,1]$ be a map such that there are $a_1, a_2 \in \mathbb{R}_{>1}$ and $c_1, c_2 \in \mathbb{R}$ such that

$$S|_{I_1}(x) = a_1 x + c_1, \qquad S|_{I_2}(x) = a_2 x + c_2, \tag{2.21}$$

$$S(I_1) = [0,1], \qquad S(I_2) = [0,1]. \tag{2.22}$$

Let Λ be the repeller for S, thus $\Lambda = \bigcap_{n=0}^{\infty} S^{-n}\left([0,1]\right)$. In [Bed91] it is proven that Theorem 2.2.14 also holds for the above defined map S. Now the Hausdorff dimension of Λ can be computed using Bowen's formula.

Let $s \in \mathbb{R}_{\geq 0}$ be such that $P(-s \log |S'|) = 0$. Let $n \in \mathbb{N}$ and $x \in I_1 \cup I_2$ be arbitrary and consider $S'(S^n x)$. If $S^n x \in I_1$, then its derivative is a_1. If $S^n x \in I_2$, its derivative is a_2. For all $y \in I_1 \cup I_2$ with $S^n y = y$ there must be a sequence $\omega = \{\omega_i\}_{i=1}^{n} \in \{1,2\}^n$ such that

$$y = S_\omega(y) := \left(S|_{I_{\omega_n}} \circ \ldots \circ S|_{I_{\omega_1}}\right)(y). \tag{2.23}$$

There are $\binom{n}{k}$ sequences in $\{1,2\}^n$ where 1 appears k times. Furthermore, S_ω has a unique fixed point and S_ω and $S_{\omega'}$ have different fixed points for all $\omega, \omega' \in \{1,2\}^n$ with $\omega \neq \omega'$. Thus

$$P\left(-t \log |S'|\right) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\substack{S^n x = x \\ x \in I_1 \cup I_2}} \left(\prod_{i=0}^{n-1} \left| S'\left(S^i x\right)\right| \right)^{-t} \right) \tag{2.24}$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\omega \in \{1,2\}^n} \prod_{i=1}^{n} a_{\omega_i}^{-t} \right) \tag{2.25}$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{k=0}^{n} \binom{n}{k} a_1^{-kt} a_2^{-t(n-k)} \right). \tag{2.26}$$

Now remember that $\sum_{k=0}^{n} \binom{n}{k} a_1^{-kt} a_2^{-t(n-k)} = \left(a_1^{-t} + a_2^{-t}\right)^n$ holds for all $n \in \mathbb{N}$ according to Newton's Binomial Theorem. Thus

$$P(-t \log |S'|) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{k=0}^{n} \binom{n}{k} a_1^{-kt} a_2^{-t(n-k)} \right) \tag{2.27}$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \left(a_1^{-t} + a_2^{-t}\right)^n \tag{2.28}$$

$$= \lim_{n\to\infty} \frac{n}{n} \log \left(a_1^{-t} + a_2^{-t}\right). \tag{2.29}$$

Therefore the Hausdorff dimension of Λ is the root of the equation $a_1^{-t} + a_2^{-t} = 1$.

The set $\Lambda = \bigcap_{n=0}^{\infty} T^{-n}\left([0,1]\right)$ is the middle-third Cantor set for the map $T\colon \left[0, \frac{1}{3}\right] \cup \left[0, \frac{2}{3}\right] \to [0,1]$ with

$$x \mapsto \begin{cases} 3x & , x \in \left[0, \frac{1}{3}\right] \\ 3x - 2 & , x \in \left[\frac{2}{3}, 1\right] \end{cases}. \tag{2.30}$$

In this case $a_1 = a_2 = 3$. Therefore the root of $2 \cdot 3^{-t} = 1$ is the dimension of the middle-third Cantor set. Since $2 \cdot 3^{-\log_3 2} = 1$, the dimension of the middle-third Cantor set is $\log_3 2$.

Chapter 3

Jenkinson-Pollicott's algorithm

Let $\mathcal{A}$ be an alphabet with $|\mathcal{A}| \geq 2$ and $T_{\mathcal{A}}$ its corresponding Gauss map as in Definition 2.1.1. In this chapter we will construct Jenkinson-Pollicott's algorithm 1 to compute the Hausdorff dimension of $E_{\mathcal{A}}$, since the methods presented in section 2.2.2 are not sufficient for this task. The Ruelle operator [Rue76] from Definition 3.1.1 in combination with Bowen's formula 2.2.14 is. The biggest eigenvalue of the Ruelle operator can be expressed in terms of pressure [Jen04]. Then $\dim_H(E_{\mathcal{A}})$ can be computed using Bowen's formula and that eigenvalue. The Ruelle operator is however too complex to compute $\dim_H(E_{\mathcal{A}})$, so Jenkinson and Pollicott [JP01, Jen04] constructed algorithm 1 on page 20 to compute the Hausdorff dimension efficiently.

In this chapter the Ruelle operator will be introduced. Then Lemma 3.1.2 will connect the Ruelle operator and pressure. Jenkinson-Pollicott's algorithm will be constructed from the Ruelle operator in section 3.2 and we also will prove that Jenkinson-Pollicott's algorithm is correct for all finite non-empty $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ with $|\mathcal{A}| \geq 2$. At last we will discuss the convergence rate and complexity of Jenkinson-Pollicott's algorithm. The conclusions, which are new results, are that the complexity is $\mathcal{O}\left(N|\mathcal{A}|^N\right)$ and the convergence rate is

$$\mathcal{O}\left(\sqrt[2|\mathcal{A}|]{\frac{4+2p}{5+2p}}^{n^2}\right) \tag{3.1}$$

where $p = \max \mathcal{A}$ and $n+1$ is the number of terms in the Taylor approximation of equation (3.3) around $z = 0$.

3.1 Ruelle operator

The Ruelle operator is introduced by Ruelle in [Rue76] and more elaborately explored in [Rue78]. Jenkinson and Pollicott studied the relation between the Ruelle operator and the Gauss map in [JP01] and [Jen04]. Their results will be heavily used in this chapter. Our definition of the Ruelle operator is taken from [JP01].

Definition 3.1.1 (Ruelle operator)**.** Let $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ be a finite and non-empty alphabet. Let $T_{\mathcal{A}}$ be the corresponding Gauss map and T_a be the a-th branch of the Gauss map with $a \in \mathcal{A}$. Let $D \subset \mathbb{C}$ be the disk $D = \left\{z \in \mathbb{C} \,\middle|\, |z-1| < \frac{3}{2}\right\}$. Let $C(D)$ be the Banach space of analytic functions on D that have a continuous extension to the boundary of D with the supremum norm. Let $s \in [0,1]$, then the *Ruelle operator* $\mathcal{L}_s \colon C(D) \to C(D)$ is defined as

$$\mathcal{L}_s v(z) = \sum_{a \in \mathcal{A}} \left(\frac{1}{z+a}\right)^{2s} v\left(\frac{1}{z+a}\right) \tag{3.2}$$

for all $v \in C(D)$ and $z \in D$.

The following lemma cited from [Jen04] explains the connection between the Ruelle operator and pressure.

Lemma 3.1.2 (Ruelle (1978), Bowen (1979))**.** *Let $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ be a finite and non-empty alphabet and $T_{\mathcal{A}}$ its corresponding Gauss map. Then the following two statements hold:*

1. *The Ruelle operator $\mathcal{L}_s$ has a simple eigenvalue $\lambda(s)$ of maximum modulus and $\lambda(s) = \exp\left(P\left(-s \log |T'_{\mathcal{A}}|\right)\right)$.*

2. *The unique solution to the equation $\lambda(s) = 1$ is $s = \dim\left(E_{\mathcal{A}}\right)$.*

Proof. The first statement is an important theorem proved by Ruelle in [Rue78] and improved by Parry and Pollicott in [PP90]. The second statement is a corollary of Theorem 2.2.14 and is proved in [Bow79]. In short, Bowen's equation $P\left(-s \log |T'|\right) = 0$ has a unique zero $s = \dim(E_{\mathcal{A}})$, thus $s = \dim\left(E_{\mathcal{A}}\right)$ is the unique solution to $\lambda(s) = \exp\left(P\left(-s \log |T'|\right)\right) = 1$. □

To compute $\dim_H\left(E_{\mathcal{A}}\right)$ we thus need to find the $s \in [0,1]$ such that $\lambda(s) = 1$ with $\lambda(s)$ from Lemma 3.1.2. This is the same as finding the $s \in [0,1]$ such that $\det\left(I - \mathcal{L}_s\right) = 0$ where I is the identity map on $C(D)$. The expression $\det\left(I - \mathcal{L}_s\right)$ can only be written in terms of infinite sums. Thus to find the zero of $s \mapsto \det\left(I - \mathcal{L}_s\right)$ we will compute the Taylor expansion of $\det\left(I - z\mathcal{L}_s\right)$ with $z \in \mathbb{C}$ around $z = 0$. This is possible since $\det\left(I - z\mathcal{L}_s\right)$ is analytic in z on D for all $s \in [0,1]$. Then we fill in $z = 1$ and find the zero of the remaining sum in s.

The function $z \mapsto \det\left(I - z\mathcal{L}_s\right)$ can be written as an exponential power, but the technique necessary to do so is outside the scope of this thesis. Ruelle [Rue76] developed this technique for the Ruelle operator and in [JP01] Jenkinson and Pollicott applied it to the case of the Gauss map. Here we will define the Fredholm determinant of the Ruelle operator as an exponential function as done in [JP01], which followed the work of [Gro95, May76, Rue76].

Definition 3.1.3. Let $s \in [0,1]$ and $z \in D$. The *Fredholm determinant* of the operator $I - z\mathcal{L}_s$ on $C(D)$ is equal to

$$\det\left(I - z\mathcal{L}_s\right) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{tr}\left(\mathcal{L}_s^n\right)\right). \tag{3.3}$$

We need to compute the trace of $\mathcal{L}_s^n$ for all $n \in \mathbb{N}_{\geq 1}$ and $s \in [0,1]$. This will be done in Lemma 3.1.10, but before we can prove Lemma 3.1.10 we need more theory on the Ruelle operator regarding its trace. In Theorem 3.1.8 we will prove that the Ruelle operator is a nuclear operator which we will define in Definition 3.1.6. This property enables us to explicitly compute the trace of the Ruelle operator. Before nuclear operators can be introduced, we need to regard trace-class operators first. The following definition is from [dG11].

Definition 3.1.4 (Trace-class operator)**.** Let H be a Hilbert space and let T be an operator on H. Let $\{e_k\}_{k\in\mathbb{N}} \subset H$ be an orthonormal basis of H. Then T is a *trace-class operator* if the following sum is finite:

$$\|T\|_1 = \text{tr } |T| := \sum_{k\in\mathbb{N}} \left\langle \sqrt{T^*T} e_k, e_k \right\rangle. \tag{3.4}$$

The following proposition from [dG11] specifies the trace of a trace-class operator.

Proposition 3.1.5. *Let H be a Hilbert space and let T be a trace-class operator on H. Let $\{e_k\}_{k\in\mathbb{N}} \subset H$ be an orthonormal basis of H. The trace of T is equal to $\text{tr } T := \sum_{k\in\mathbb{N}} \langle Te_k, e_k\rangle$ and it is absolutely convergent and independent of choice of basis.*

Thus trace-class operators have a trace that is independent on the choice of basis and Proposition 3.1.5 gives us an explicit formula for the trace of a trace-class operator. It is not easy to prove by definition that the Ruelle operator is a trace-class operator. Grothendieck [Gro95] introduced nuclear operators as a special kind of trace-class operators.

Definition 3.1.6 (Nuclear)**.** Let $L\colon C(D) \to C(D)$ be a linear operator with $C(D)$ as in Definition 3.1.1. The operator L is *nuclear* if there are a sequence of functions $\{f_n\}_{n=1}^{\infty} \subset C(D)$ with $\|f_n\| = 1$, a sequence of normalized functionals $\{l_n\}_{n=1}^{\infty} \subset C(D)^*$ where $C(D)^*$ is the dual space of $C(D)$, and a sequence of complex coefficients $\{\lambda_n\}_{n=1}^{\infty} \in \ell^1(\mathbb{C})$ such that $Lv = \sum_{n=1}^{\infty} (\lambda_n l_n(v)) f_n$ holds for all $v \in C(D)$.

Remark 3.1.7. Grothendieck [Gro95] proved that a nuclear operator is compact.

Theorem 3.1.8 proves that $\mathcal{L}_s^n$ is a nuclear operator for all $s \in [0,1]$ and $n \in \mathbb{N}_{\geq 1}$. This proof draws heavily from [JP01] where Theorem 3.1.8 and Lemma 3.1.10 are proven with $\mathcal{A} = \{1, 2\}$.

Theorem 3.1.8. *Let $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ be an alphabet with $|\mathcal{A}| \geq 2$, then the Ruelle operator $\mathcal{L}_s$ is nuclear for all $s \in [0,1]$.*

Proof. Consider the disk $D = \left\{ z \in \mathbb{C} \,\middle|\, |z-1| < \frac{3}{2} \right\}$ as in Definition 3.1.1. First we will compute $T_a(D)$ for all $a \in \mathcal{A}$ where T_a from Definition 2.1.1 is extended to $T_a\colon D \to \mathbb{C}$. The function T_a can for all $a \in \mathcal{A}$ be specified as a Möbius transformation $T_a(z) = \frac{1}{z+a}$ for all $z \in D$. One property of Möbius transformation is that generalized circles are invariant, where a generalized circle is a circle or a line [FB09]. In our case $T_a(D)$ will either be a disk or a combination of circles and lines. Since $-a \notin D$ holds for all $a \in \mathcal{A}$, there will be no lines to infinity in $T_a(D)$. Thus $T_a(D)$ is a disk for all $a \in \mathcal{A}$.

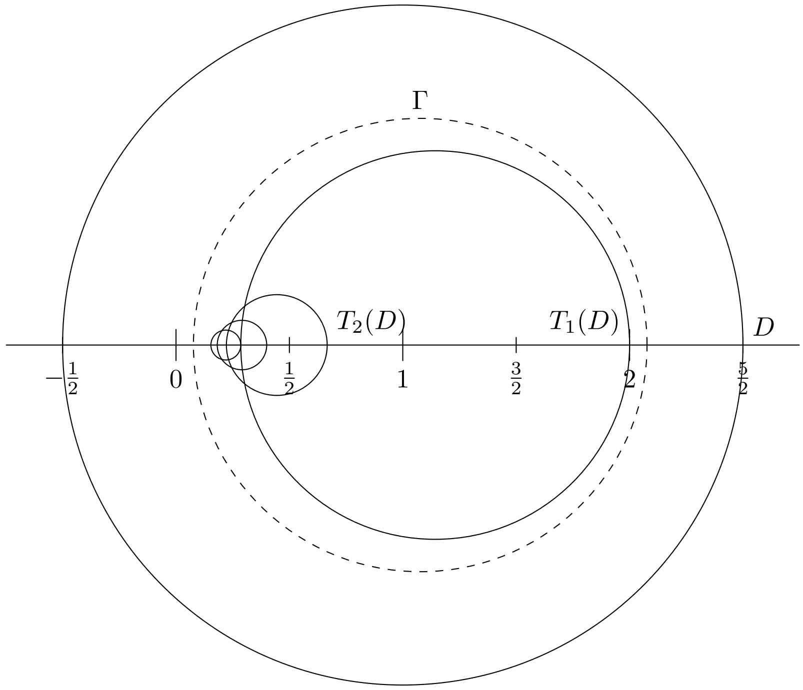

Figure 3.1: Visualization of $T_a(D)$ for $a \in \{0, 1, 2, 3, 4\}$ with the curve Γ. Note that the disks $T_a(D)$ are smaller if a increases.

Let $a \in \mathcal{A}$ be arbitrary and consider $T_a\left(\left(-\frac{1}{2}, \frac{5}{2}\right)\right)$. The map T_a is strictly decreasing and positive on $\left(-\frac{1}{2}, \frac{5}{2}\right)$. The end points of $\left(-\frac{1}{2}, \frac{5}{2}\right)$ are mapped to $T_a\left(-\frac{1}{2}\right) = \frac{2}{2a-1}$ and $T_a\left(\frac{5}{2}\right) = \frac{2}{5+2a}$, which gives

$$T_a\left(\left(-\frac{1}{2}, \frac{5}{2}\right)\right) = \left(\frac{2}{5+2a}, \frac{2}{2a-1}\right). \tag{3.5}$$

Since T_a is strictly decreasing, the points $\frac{2}{5+2a}$ and $\frac{2}{2a-1}$ are two points in $T_a(D)$ with maximal distance. This gives us of the disk $T_a(D)$ the center $\frac{1}{2}\left(\frac{2}{5+2a} + \frac{2}{2a-1}\right) = \frac{4a+4}{(5+2a)(2a-1)}$ and radius $\frac{4a+4}{(5+2a)(2a-1)} - \frac{2}{5+2a} = \frac{6}{(5+2a)(2a-1)}$. Thus

$$T_a(D) = \left\{ z \in \mathbb{C} \,\middle|\, \left| z - \frac{4a+4}{(5+2a)(2a-1)} \right| < \frac{6}{(5+2a)(2a-1)} \right\}. \tag{3.6}$$

In figure 3.1 a visualization of $T_a(D)$ is given for some $a \in \mathbb{N}_{\geq 1}$. Note that the visualization given in [JP01] may not be correct, as the written and drawn $T_1(D)$ in figure 1 in [JP01] clearly are not the same.

Let $p = \max \mathcal{A}$. We need to find a circle $\Gamma \subset D$ of radius 1 that encloses $T_a(D)$ for all $a \in \mathcal{A}$ to apply Cauchy's integral formula later. Note that

$$\min_{a \in \mathcal{A}} \min_{z \in T_a(D)} \operatorname{Re}(z) = \min_{z \in T_p(D)} \operatorname{Re}(z) = \frac{2}{5+2p}. \tag{3.7}$$

A circle that passes though $\frac{1}{5+2p}$ and $\frac{11+4p}{5+2p}$ will do, which is the curve

$$\Gamma = \left\{ z \in \mathbb{C} \;\middle|\; \left| z - \frac{6+2p}{5+2p} \right| = 1 \right\}. \tag{3.8}$$

Let $\Gamma' = \left\{ z \in \mathbb{C} \;\middle|\; \left| z - \frac{6+2p}{5+2p} \right| < 1 \right\}$ be the interior disk of Γ. Let $s \in [0,1]$. Let $v \colon \Gamma' \to \mathbb{C}$ be a holomorphic function. The function $\xi \mapsto \xi^{2s}$ is well defined and analytic on Γ'. Thus Cauchy's integral formula can be used to write

$$(T_a z)^{2s}\, v(T_a z) = \frac{1}{2\pi i} \int_\Gamma \frac{\xi^{2s} v(\xi)}{\xi - T_a z} d\xi \tag{3.9}$$

for all $z \in \Gamma'$ and $a \in \mathcal{A}$. For all $z \in \Gamma'$ the following sum holds:

$$\begin{aligned}
\mathcal{L}_s v(z) &= \sum_{a \in \mathcal{A}} (T_a z)^{2s}\, v(T_a z) &(3.10)\\
&= \frac{1}{2\pi i} \int_\Gamma \xi^{2s} v(\xi) \sum_{a \in \mathcal{A}} \frac{1}{\xi - T_a z} d\xi &(3.11)\\
&= \frac{1}{2\pi i} \int_\Gamma \frac{\xi^{2s} v(\xi)}{\xi - \frac{6+2p}{5+2p}} \sum_{a \in \mathcal{A}} \left(1 - \frac{T_a z - \frac{6+2p}{5+2p}}{\xi - \frac{6+2p}{5+2p}} \right)^{-1} d\xi &(3.12)\\
&= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_\Gamma \frac{\xi^{2s} v(\xi)}{\left(\xi - \frac{6+2p}{5+2p}\right)^{n+1}} d\xi \sum_{a \in \mathcal{A}} \left(T_a z - \frac{6+2p}{5+2p} \right)^n &(3.13)
\end{aligned}$$

Note that the expansion in infinite sum is true since $T_a z \in \Gamma'$ for all $a \in \mathcal{A}$ and $z \in D$ gives

$$\left| \frac{T_a z - \frac{6+2p}{5+2p}}{\xi - \frac{6+2p}{5+2p}} \right| < 1 \tag{3.14}$$

for all $z \in D$, $a \in \mathcal{A}$ and $\xi \in \Gamma$. Now, following the notation of [JP01], define

$$g_{a,n}(z) := \left(T_a z - \frac{6+2p}{5+2p} \right)^n, \quad m_n(v) := \frac{1}{2\pi i} \int_\Gamma \frac{\xi^{2s} v(\xi)}{\left(\xi - \frac{6+2p}{5+2p}\right)^{n+1}} d\xi, \tag{3.15}$$

$$f_{a,n} := \frac{g_{a,n}}{\|g_{a,n}\|}, \qquad l_n := \frac{m_n}{\|m_n\|}. \tag{3.16}$$

for all $a \in \mathcal{A}$, $n \in \mathbb{N}$, $z \in D$ and holomorphic $v \colon \Gamma' \to \mathbb{C}$. Lastly define for all $a \in \mathcal{A}$ and $n \in \mathbb{N}$ the constants $\lambda_{a,n} = \|g_{a,n}\| \, \|m_n\|$, then $\mathcal{L}_s$ can be expressed as

$$\mathcal{L}_s v(z) = \sum_{n=0}^{\infty} \sum_{a \in \mathcal{A}} (\lambda_{a,n} l_n(v))\, f_{a,n}(z). \tag{3.17}$$

We now only need to prove $\sum_{n=0}^{\infty} \sum_{a \in \mathcal{A}} \lambda_{a,n} < \infty$. For that we need to estimate $\|g_{a,n}\|$ and $\|m_n\|$ for all $a \in \mathcal{A}$ and $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be arbitrary and consider $\|g_{p,n}\|$ first. Note that $p = 1$ cannot hold since $|\mathcal{A}| \geq 2$. The set $T_p(D)$ is a disk not containing $\frac{6+2p}{5+2p}$ and $T_p(D)$ is located left of $\frac{6+2p}{5+2p}$. This gives

$$\|g_{p,n}\| = \left| \min_{z \in D} \operatorname{Re} T_p(z) - \frac{6+2p}{5+2p} \right|^n = \left| \frac{2}{5+2p} - \frac{6+2p}{5+2p} \right|^n = \left(\frac{4+2p}{5+2p} \right)^n. \tag{3.18}$$

As can be seen in figure 3.1 the regions $T_a(D)$ are for all $a \in \mathcal{A} \setminus \{p\}$ closer to $\frac{6+2p}{5+2p}$. Thus $\|g_{a,n}\| \leq \left(\frac{4+2p}{5+2p}\right)^n$ holds for all $a \in \mathcal{A}$ and $n \in \mathbb{N}$.
Let $n \in \mathbb{N}$ and consider $\|m_n\|$. Let $v\colon \Gamma' \to \mathbb{C}$ be holomorphic such that $\|v\| = 1$ holds. For all $\xi \in \Gamma$ the equality $\xi - \frac{6+2p}{5+2p} = 1$ is per definition of Γ true. Furthermore, $|\xi| \leq \frac{11+4p}{5+2p}$ gives $|\xi|^{2s} \leq 5$ for all $s \in [0,1]$. The length of curve Γ is 2π since Γ is a circle of radius 1. This gives for all $n \in \mathbb{N}$ the following estimation of $\|m_n\|$:

$$\|m_n\| = \sup_{\|v\|\leq 1} \left| \frac{1}{2\pi i} \int_\Gamma \frac{\xi^{2s} v(\xi)}{\left(\xi - \frac{6+2p}{5+2p}\right)^{n+1}} d\xi \right| \leq \frac{2\pi}{2\pi} \frac{5 \cdot 1}{1^{n+1}} = 5. \tag{3.19}$$

Now the following exponential upper bound $\lambda_{a,n} = \|g_{a,n}\| \, \|m_n\| \leq 5 \left(\frac{4+2p}{5+2p}\right)^n$ arises. This means that the sum $\sum_{n=0}^{\infty} \sum_{a\in\mathcal{A}} \lambda_{a,n}$ is finite, which proves that $\mathcal{L}_s$ is nuclear. □

The upper bound for the $\lambda_{a,n}$ in the proof above will in section 3.2.3 be used to compute the convergence of Jenkinson-Pollicott's algorithm. Grothendieck's propostion [Gro95] applied in [JP01] provides the trace of $\mathcal{L}_s^n$.

Proposition 3.1.9 (Grothendieck (1995)). *The zeros of $z \mapsto \det(I - z\mathcal{L}_s)$ are the non-zero eigenvalues of $\mathcal{L}_s$ with both zeros counted with multiplicities.*

There is now enough theory to explicitly compute the trace of the Ruelle operator. The proof of Lemma 3.1.10 is in [JP01], but that proof is for $\mathcal{A} = \{1,2\}$ only. However, since for all finite $\mathcal{A} \subset \mathbb{N}_{\geq 2}$ with $|\mathcal{A}| \geq 2$ the Ruelle operator is nuclear and since Jenkinson and Pollicott did not explicitly use the equality $\mathcal{A} = \{1,2\}$, their proof also applies for Lemma 3.1.10.

Lemma 3.1.10 (Jenkinson, Pollicott (2001)). *Let $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ be a finite alphabet with $|\mathcal{A}| \geq 2$. Let $s \in [0,1]$ and $n \in \mathbb{N}_{\geq 1}$. Let $\underline{i} \in \mathcal{A}^n$ be a finite string. Let $|\underline{i}| \in \mathbb{N}$ be equal to the number of elements in $\underline{i}$. Let $x_{\underline{i}} \in E_{\mathcal{A}}$ be the periodic continued fraction $x_{\underline{i}} = [i_1, \ldots, i_n, i_1, \ldots, i_n, i_1, \ldots]$. Let $w_{\underline{i}} \in \mathbb{R}_{\geq 0}$ be the weight of $x_{\underline{i}}$ defined as $w_{\underline{i}} = \prod_{r=0}^{|\underline{i}|-1} T_{\mathcal{A}}^r\left(x_{\underline{i}}\right)$. Then the trace of $\mathcal{L}_s^n$ is equal to*

$$\operatorname{tr}\left(\mathcal{L}_s^n\right) = \sum_{\substack{\underline{i}\in\mathcal{A}^n \\ |\underline{i}|=n}} \frac{w_{\underline{i}}^{2s}}{1 - (-1)^{|\underline{i}|} w_{\underline{i}}^2}. \tag{3.20}$$

Take a look at the definition of the weight $w_{\underline{i}}$ of $x_{\underline{i}}$ for $\underline{i} \in \mathcal{A}^n$. Define for all $n \in \mathbb{N}_{\geq 1}$ the concepts $\underline{i}$, $x_{\underline{i}}$ and $w_{\underline{i}}$ for the rest of the thesis as defined in Lemma 3.1.2. Let $\{X_i\}_{i=1}^k \subset [0,1]$ be a family of closed disjoint intervals and let $X := \bigcup_{i=1}^k X_i \subset [0,1]$. Let $S\colon X \to [0,1]$ be such that for all $i \in \{1,\ldots,k\}$ the map $S|_{X_i}$ is continuous and bijective. The Fixed Point Theorem states that for all $i \in \{1,\ldots,k\}$ the map $S|_{X_i}$ has a fixed point. Let $n \in \mathbb{N}$ and let $x \in [0,1]$ be a fixed point of S^n. The weight $w \in \mathbb{R}_{\geq 0}$ of a fixed point x of S^n is defined as $w = \prod_{r=0}^n |S'(S^r(x))|^{-1}$. We will now derive that our definition of weight is equal to the general definition.

Let $n \in \mathbb{N}$, let $\underline{i} \in \mathcal{A}^n$ and let $x_{\underline{i}} \in [0,1]$ be a fixed point of $T_{\mathcal{A}}^{|\underline{i}|-1}$ where $T_{\mathcal{A}}$ is the Gauss map with restricted digits. Denote $x_{\underline{i}} = [a_0, a_1, \ldots]$ with $\{a_i\}_{i\in\mathbb{N}} \subset \mathcal{A}$

and recall that for the Gauss map $T_{\mathcal{A}}([a_0, a_1, \ldots]) = [a_1, a_2, \ldots]$ holds. The weight of $x_{\underline{i}}$ is equal to

$$w = \prod_{r=0}^{|\underline{i}|-1} \left| T'_{\mathcal{A}}\left(T^r_{\mathcal{A}}(x_{\underline{i}})\right)\right|^{-1} = \prod_{r=0}^{|\underline{i}|-1} \left| T'_{\mathcal{A}}\left([a_r, a_{r+1}, \ldots]\right)\right|^{-1} \tag{3.21}$$

$$= \prod_{r=0}^{|\underline{i}|-1} \left| -\frac{1}{[a_r, a_{r+1}, \ldots]^2} \right|^{-1} = \prod_{r=0}^{|\underline{i}|-1} [a_r, a_{r+1}, \ldots]^2 \tag{3.22}$$

$$= \prod_{r=0}^{|\underline{i}|-1} \left(T^r_{\mathcal{A}}\left(x_{\underline{i}}\right)\right)^2 = w_{\underline{i}}^2. \tag{3.23}$$

Thus in our case the weight $w_{\underline{i}}$ of a fixed point $x_{\underline{i}}$ is well defined for all $\underline{i} \in \mathcal{A}^n$ with $|\underline{i}| = n$.

This section concludes to the now fully established definition of $\det(I - z\mathcal{L}_s)$:

$$\det(I - z\mathcal{L}_s) = \exp\left(-\sum_{n=1}^{\infty} \sum_{\substack{\underline{i} \in \mathcal{A}^n \\ |\underline{i}|=n}} \frac{1}{n} \frac{w_{\underline{i}}^{2s}}{1 - (-1)^{|\underline{i}|}\, w_{\underline{i}}^2} \right). \tag{3.24}$$

3.2 Jenkinson-Pollicott's algorithm

Lemma 3.1.2 states that we need to compute the zero of $s \mapsto \det(I - \mathcal{L}_s)$ with $s \in [0, 1]$. Jenkinson and Pollicott [JP01, Jen04] constructed an algorithm, here algorithm 1, for finding this zero. The construction of algorithm 1 relies heavily on equation (3.24). In this section we first construct algorithm 1 ourselves and prove that algorithm 1 is correct for all finite $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ with $|\mathcal{A}| \geq 2$. In sections 3.2.2 and 3.2.3 we will consider respectively the complexity and convergence rate of algorithm 1.

Remark 3.2.1. Equation (3.24) is an exponential power, which can never be 0. However, equation (3.24) is not defined everywhere. There is another definition from [Rue76] for Fredholm determinant which is more complicated but is defined on a larger domain than equation (3.24). Let $\Lambda^k C(D)$ be the k-th exterior power for the complex Hilbert space $C(D)$ for all $k \in \mathbb{N}$ with $C(D)$ as in Definition 3.1.1. Define for all linear maps $A \colon C(D) \to C(D)$ and $k \in \mathbb{N}$ the bounded operator $\Lambda^k(A)$ on $\Lambda^k C(D)$ by

$$\Lambda^k(A)\,(v_1 \wedge \ldots \wedge v_k) = Av_1 \wedge \ldots \wedge Av_k \tag{3.25}$$

with $v_i \in C(D)$ for all $i \in \{1, \ldots, k\}$. The Fredholm determinant for the operator $I - z\mathcal{L}_s$ can for all $z \in \mathbb{C}$ and $s \in [0, 1]$ also be defined as

$$\det(I - z\mathcal{L}_s) = \sum_{k=0}^{\infty} (-1)^k \mathrm{tr}\left(\Lambda^k\left(\mathcal{L}_s\right)\right). \tag{3.26}$$

This definition is equal to equation (3.24) on the intersection of their domains. For algorithm 1 we keep using equation 3.24 because algorithm 1 returns the desired zero and it is easier to manipulate since an explicit formula is known.

The computations used to compute the zero of $s \mapsto \det(I - \mathcal{L}_s)$ are shown in algorithm 1, also called Jenkinson-Pollicott's algorithm.

Algorithm 1 Jenkinson-Pollicott's algorithm.

Choose a $N \in \mathbb{N}$.
for all $n \in \{1, \dots, N\}$ **do**
  Compute all strings $\underline{i} \in \mathcal{A}^n$ with $|\underline{i}| = n$.
  for all $\underline{i} \in \mathcal{A}^n$ **do**
    Compute $x_{\underline{i}} = [i_1, \dots, i_n, i_1, \dots, i_n, i_1, \dots]$.
    Compute $w_{\underline{i}} = \prod_{r=0}^{n-1} T^r \left(x_{\underline{i}}\right)$.
    Define $F_{\underline{i}}(s) = \frac{w_{\underline{i}}^{2s}}{1-(-1)^n w_{\underline{i}}^2}$.
  end for
  Define $c_n(s) = \frac{1}{n} \sum_{\substack{\underline{i} \in \mathcal{A}^n \\ |\underline{i}|=n}} F_{\underline{i}}(s)$.
  Define $d_n(s) = \sum_{j=1}^{n} \sum_{\substack{m \in \mathbb{N}_{\geq 1}^j \\ \sum_{l=1}^{j} m_l = n}} \frac{(-1)^j}{j!} \prod_{l=1}^{j} c_{m_l}(s)$.
end for
Define $\Delta_N(s) = 1 + \sum_{n=1}^{N} d_n(s)$.
  ▷ This $\Delta_N(s)$ is a Taylor polynomial of order N of $s \mapsto \det(I - \mathcal{L}_s)$.
Compute the largest zero $s = s_N$ of $s \mapsto \Delta_N(s)$.
  ▷ In this thesis algorithm 2 is used.
return s_N.

3.2.1 Correctness

We will now construct algorithm 1. Let $s_0 \in [0, 1]$ be the zero of the function $s \mapsto \det(I - \mathcal{L}_s)$. In order to find s_0 we need to solve equation (3.24). Solving equation (3.24) analytically is nearly impossible. The solution of equation (3.24) can be approximated by the analytic continuation of $z \mapsto \det(I - z\mathcal{L}_s)$ on D as stated in [JP01]. We can compute $\det(I - \mathcal{L}_s)$ by constructing the Taylor polynomial of $\det(I - z\mathcal{L}_s)$ around $z = 0$ and fill in $z = 1$.
The Taylor polynomial of e^z around $z = 0$ is given by $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. This gives the following Taylor polynomial of $\det(I - z\mathcal{L}_s)$:

$$\det(I - z\mathcal{L}_s) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} \left(\sum_{n=1}^{\infty} \sum_{\substack{\underline{i} \in \mathcal{A}^n \\ |\underline{i}|=n}} \frac{1}{n} \frac{w_{\underline{i}}^{2s}}{1 - (-1)^{|\underline{i}|} w_{\underline{i}}^2} \right)^k . \tag{3.27}$$

Define $c_n(s) = \sum_{\substack{\underline{i} \in \mathcal{A}^n \\ |\underline{i}|=n}} \frac{1}{n} \frac{w_{\underline{i}}^{2s}}{1-(-1)^{|\underline{i}|} w_{\underline{i}}^2}$ for all $n \in \mathbb{N}_{\geq 1}$. This simplifies equation (3.27) to $\det(I - z\mathcal{L}_s) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} \left(\sum_{n=1}^{\infty} c_n(s)\right)^k$. First we will analyze $\left(\sum_{n=1}^{\infty} c_n(s)\right)^k$. In that expression the $c_{n_j}(s)$'s are chosen with $n_j \in \mathbb{N}_{\geq 1}$ for all $j \in \{1, \dots, k\}$. All those terms are then multiplied, which gives us

$$\det(I - z\mathcal{L}_s) = 1 + \sum_{k=1}^{\infty} \frac{(-z)^k}{k!} \sum_{m \in \mathbb{N}_{\geq 1}^k} \prod_{l=1}^{k} c_{m_l}(s). \tag{3.28}$$

Now we must choose between two options. The first option is to truncate the first sum of equation (3.28) to N terms with $N \in \mathbb{N}_{\geq 1}$. However, the sum

$\sum_{m\in\mathbb{N}^k_{\geq 1}} \prod_{l=1}^{j} c_{m_l}(s)$ will still be an infinite sum, so for the approximation we must truncate N other infinite sums. This will lead to estimation errors that in this case are not easy to estimate and control.
The second option is to set $N \in \mathbb{N}_{\geq 1}$ and consider for all $k \in \mathbb{N}_{\geq 1}$ only the $m \in \mathbb{N}^k_{\geq 1}$ where the terms of m sum up to an $n \in \{1, \dots, N\}$. In this way the sum $\sum_{m\in\mathbb{N}^k_{\geq 1}} \prod_{l=1}^{k} c_{m_l}(s)$ will be truncated for all $k \in \mathbb{N}_{\geq 1}$. Then we can truncate the sum the first sum of equation (3.28) to N terms to get our desired approximation since term $N+1$ and above do not exist anymore due to the earlier truncation. This thesis chose the second option.

Let $N \in \mathbb{N}_{\geq 1}$ be arbitrary. Then the approximation will be as following:

$$\det(I - z\mathcal{L}_s) = 1 + \sum_{k=1}^{\infty} \sum_{m\in\mathbb{N}^k_{\geq 1}} \frac{(-z)^k}{k!} \prod_{l=1}^{k} c_{m_l}(s) \tag{3.29}$$

$$\approx 1 + \sum_{n=1}^{N} \sum_{j=1}^{n} \sum_{\substack{m\in\mathbb{N}^j_{\geq 1} \\ \sum_{k=1}^{j} m_k = n}} \frac{(-z)^j}{j!} \prod_{l=1}^{j} c_{m_l}(s) \tag{3.30}$$

This gives us the $\Delta_N(s)$ of algorithm 1. We also proved that $\Delta_N(s)$ is an approximation of equation (3.27). The error of this approximation will be considered in section 3.2.3.

When $\Delta_N(s)$ is constructed in algorithm 1 one must compute the largest zero $s = s_N$ of $s \mapsto \Delta_N(s)$. Various methods can be used to compute the largest zero. Note that $s \mapsto \det(I - \mathcal{L}_s)$ is a strictly increasing function on $[0,1]$, thus $s \mapsto \Delta_N(s)$ must be strictly increasing on $[0,1]$ as well since it is a Taylor approximation. Thus the bisection algorithm, here algorithm 2, can be used to compute this zero.

Algorithm 2 Bisection algorithm for $s \mapsto \Delta_N(s)$

Define $a \leftarrow 0$ and $b \leftarrow 1$.
Define $s' \leftarrow \frac{a+b}{2} = \frac{1}{2}$.
while $|a - b| > 10^{-8}$ **do**
  Compute $\Delta_N(s')$.
  if $\Delta_N(s') < 0$ **then**
    Define $a \leftarrow s'$.
    Define $s' \leftarrow \frac{a+b}{2}$.
  else
    Define $b \leftarrow s'$.
    Define $s' \leftarrow \frac{a+b}{2}$.
  end if
end while
return s' ▷ This s' is an approximation for $\dim_H(E_{\mathcal{A}})$.

Algorithm 2 is constructed in the following way. Let $s_0 \in [0,1]$ be the zero of $s \mapsto \Delta_N(s)$. Let $a = 0$ and $b = 1$. Since $s \mapsto \Delta_N(s)$ is strictly increasing and since we must find s_0, $\Delta_N(a) \leq 0$ and $\Delta_N(b) \geq 0$ must hold. Let $s' = \frac{1}{2}$ and compute $\Delta_N\left(\frac{1}{2}\right)$. If $\Delta_N\left(\frac{1}{2}\right) < 0$ is negative, then $s_0 \in (s', b]$. Define then

$a := s'$ and $s' := \frac{s'+b}{2}$ and compute $\Delta_N(s')$ again. If $\Delta_N\left(\frac{1}{2}\right) > 0$ is positive, then $s_0 \in [a, s')$ holds. Define then $b := s'$ and $s' := \frac{a+s'}{2}$ and compute $\Delta_N(s')$ again. This will almost never lead to the exact s', however $|a-b|$ will decrease in every step by factor $\frac{1}{2}$. One can set a required upper bound for $|a-b|$ when the algorithm needs to terminate. In this thesis the upper bound 10^{-8} is chosen.

We have now proven the following theorem:

Theorem 3.2.2. *Algorithm 1 is correct for finite $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ with $|\mathcal{A}| \geq 2$.*

Proof. From section 3.2 onward, we have found the approximation

$$\det(I - z\mathcal{L}_s) \approx 1 + \sum_{n=1}^{N} \sum_{j=1}^{n} \sum_{\substack{m \in \mathbb{N}_{\geq 1}^{j} \\ \sum_{l=1}^{j} m_l = n}} \frac{(-z)^j}{j!} \prod_{l=1}^{j} \sum_{\substack{\underline{i} \in \mathcal{A}^{m_l} \\ |\underline{i}| = m_l}} \frac{1}{m_l} \frac{\prod_{r=0}^{|\underline{i}|-1} \left(T^r\left(x_{\underline{i}}\right)\right)^{2s}}{1 - (-1)^{|\underline{i}|} \prod_{r=0}^{|\underline{i}|-1} \left(T^r\left(x_{\underline{i}}\right)\right)^2} \tag{3.31}$$

with $N \in \mathbb{N}_{\geq 1}$. Lemma 3.1.2 states that the largest zero of $s \mapsto \det(I - \mathcal{L}_s)$ needs to be computed. This zero can be approximated by filling in $z = 1$ in the approximation, which gives $\Delta_N(s)$ of algorithm 1, and then computing the largest zero of $s \mapsto \Delta_N(s)$. This proves the correctness of algorithm 1. □

3.2.2 Complexity of Jenkinson-Pollicott's algorithm

Now we know that Jenkinson-Pollicott's algorithm is correct, we ask ourselves two questions: how long will it take for Jenkinson-Pollicott's algorithm to terminate and how fast does the returned value of Jenkinson-Pollicott's algorithm converge to $\dim_H(E_{\mathcal{A}})$? The first question will be treated in this section, the second one in section 3.2.3.
To answer the first question we will treat algorithm 1 line by line and count the number of steps in every line to find a complexity of $\mathcal{O}(n|\mathcal{A}|^n)$ where $n+1$ is the number of terms used in the Taylor approximation of $\det(I - \mathcal{L}_s)$. This will give us an upper bound for the number of steps needed.

Theorem 3.2.3. *The complexity of algorithm 1 is $\mathcal{O}\left(N|\mathcal{A}|^N\right)$ where N is the number of terms in the Taylor approximation of $\det(I - \mathcal{L}_s)$.*

Proof. Let $N \in \mathbb{N}_{\geq 1}$ be arbitrary. Consider algorithm 1. We will now estimate the complexity of every line in algorithm 1 and combine all the complexities to deduct the complexity of the whole algorithm.

We start with line 3. Let $n \in \{1, \ldots, N\}$ and $\underline{i} \in \mathcal{A}^n$ with $|\underline{i}| = n$ be arbitrary. Then $\underline{i}$ is a sequence of n components with $|\mathcal{A}|$ possible letters for every element in the sequence. There are $|\mathcal{A}|^n$ different $\underline{i} \in \mathcal{A}^n$ with $|\underline{i}| = n$. In total line 3 computes $\sum_{n=1}^{N} |\mathcal{A}|^n$ sequences. Thus the complexity of line 3 as $\mathcal{O}\left(|\mathcal{A}|^N\right)$.

Consider now lines 4 to 8. Let $n \in \{1, \ldots, N\}$ be arbitrary and $\underline{i} \in \mathcal{A}^n$ with $|\underline{i}| = n$ be arbitrary. To compute $x_{\underline{i}}$ we start with the continued fraction $[i_n]$. Then $\left(T_{i_1} \circ \cdots \circ T_{i_{n-1}}\right)([i_n]) = [i_1, \ldots, i_n]$ is computed. For better numerical approximation we compute $\left(T_{i_1} \circ \cdots \circ T_{i_n}\right)^{k-1}([i_1, \ldots, i_n])$ with $k \in \mathbb{N}_{\geq 1}$. In total we need kn steps to compute $x_{\underline{i}}$ where $k \in \mathbb{N}_{\geq 1}$ is the number of sequences

$i_1, \ldots, i_n$ in the partial continued fraction. There are $|\mathcal{A}|^n$ different $\underline{i} \in \mathcal{A}^n$ with $|\underline{i}| = n$, thus $kn|\mathcal{A}|^n$ steps will be taken to compute all $x_{\underline{i}}$ with $|\underline{i}| = n$. This concludes that line 5 takes $\sum_{n=1}^{N} kn|\mathcal{A}|^n = \mathcal{O}\left(N|\mathcal{A}|^N\right)$ steps.
If we want to compute $w_{\underline{i}}$, we need to apply the Gauss map n times on $x_{\underline{i}}$. There are n different $x_{\underline{i}}$ with $|\underline{i}| = n$, thus line 6 needs $\sum_{n=1}^{N} n|\mathcal{A}|^n = \mathcal{O}\left(N|\mathcal{A}|^N\right)$ steps to be executed. In line 7 there are in total $\sum_{n=1}^{N} |\mathcal{A}|^n = \mathcal{O}\left(|\mathcal{A}|^N\right)$ functions defined. Thus the complexity of lines 4 to 8 is $\mathcal{O}\left(N|\mathcal{A}|^N\right)$ due to lines 5 and 6.

Consider now line 9. Let $n \in \{1, \ldots, N\}$ be arbitrary. For every $c_n(s)$ we need $|\mathcal{A}|^n$ functions $F_{\underline{i}}(s)$ and 1 step extra for the $\frac{1}{n}$. Thus to execute line 9 one time we need $|\mathcal{A}|^n + 1$ steps, thus $\sum_{n=1}^{N} |\mathcal{A}|^n + N = \mathcal{O}\left(|\mathcal{A}|^N\right)$ steps in total.

Line 10 is more complicated. Let $n \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, n\}$ be arbitrary. For every $m = (m_1, \ldots, m_j) \in \mathbb{N}_{\geq 1}^j$ with $\sum_{l=1}^{j} m_l = n$ we need $j+1$ multiplications to compute $\frac{(-1)^j}{j!} \prod_{l=1}^{j} c_{m_l}(s)$. Now we need to count the number of $m \in \mathbb{N}_{\geq 1}^j$ with $\sum_{l=1}^{j} m_l = n$.
The number of $m \in \mathbb{N}_{\geq 1}^j$ with $\sum_{l=1}^{j} m_l = n$ is the same as the coefficient of x^n in $\left(\sum_{i=1}^{\infty} x^i\right)^j = \frac{x^j}{(1-x)^j}$. Recall the standard sum $(1-x)^{-j} = \sum_{i=0}^{\infty} \binom{j+i-1}{j-1} x^i$. This gives us the expression $\left(\sum_{i=1}^{\infty} x^i\right)^j = \sum_{i=0}^{\infty} \binom{j+i-1}{j-1} x^{i+j}$. Filling in index $i = n-j$ gives $\binom{n-1}{j-1}$ as the coefficient of x^n.
Thus there are $\binom{n-1}{j-1}$ different $m \in \mathbb{N}_{\geq 1}^j$ with $\sum_{l=1}^{j} m_l = n$. We need $(j+1)\binom{n-1}{j-1}$ steps to compute

$$\sum_{\substack{m \in \mathbb{N}_{\geq 1}^j \\ \sum_{l=1}^{j} m_l = n}} \frac{(-1)^j}{j!} \prod_{l=1}^{j} c_{m_l}(s). \tag{3.32}$$

To execute line 10 we need $\sum_{j=1}^{n} (j+1)\binom{n-1}{j-1}$ steps for every $n \in \{1, \ldots, N\}$, thus for the whole execution we need $\sum_{n=1}^{N} \sum_{j=1}^{n} (j+1)\binom{n-1}{j-1}$ steps in total. This concludes that line 10 has complexity $\mathcal{O}\left(N \cdot \binom{N-1}{\lfloor \frac{N-1}{2} \rfloor}\right)$.
Suppose $|\mathcal{A}| = 2$ holds and compare $\mathcal{O}\left(N \cdot \binom{N-1}{\lfloor \frac{N-1}{2} \rfloor}\right)$ and $\mathcal{O}\left(N2^N\right)$. The sum $2^N = \sum_{k=0}^{N} \binom{N}{k}$ gives us the estimation $N \cdot \binom{N-1}{\lfloor \frac{N-1}{2} \rfloor} = \mathcal{O}\left(N2^{N-1}\right)$. Thus the complexity of line 10 is less than $\mathcal{O}\left(N2^N\right)$. This is less than the bound $\mathcal{O}\left(N\,|\mathcal{A}|^N\right)$ where $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ is a general finite alphabet with $|\mathcal{A}| \geq 2$.

Line 12 needs $N+1$ steps, thus has complexity $\mathcal{O}(N+1)$. For line 14 one can choose his own algorithm to compute the zero of $s \mapsto \Delta_N(s)$. The bisection algorithm 2 chosen here has complexity smaller than $\mathcal{O}\left(N|\mathcal{A}|^N\right)$.

To conclude, lines 5 and 6 have the largest complexity, namely $\mathcal{O}\left(N|\mathcal{A}|^N\right)$. Thus the complexity of algorithm 1 is $\mathcal{O}\left(N|\mathcal{A}|^N\right)$. □

3.2.3 Convergence

In section 3.2.2 we found that the complexity of the algorithm is $\mathcal{O}\left(n|\mathcal{A}|^n\right)$ where $n+1$ is the number of terms in the Taylor approximation of $\det\left(I - z\mathcal{L}_s\right)$

around $z = 0$. This means that adding a term to the Taylor approximation rapidly increases the computation time. We want to find the rate of convergence of s_N, the output of algorithm 1, to $\dim_H(E_{\mathcal{A}})$. Jenkinson [Jen04] stated that this rate is $\mathcal{O}\left(\theta^{n^2}\right)$ with $\theta \in (0,1)$ and in [JP01] they proved that for $\mathcal{A} = \{1,2\}$ the rate is

$$\mathcal{O}\left(\sqrt[4]{\frac{8}{9}}^{n^2}\right). \tag{3.33}$$

In this section we will prove that the rate of convergence is

$$\mathcal{O}\left(\sqrt[2|\mathcal{A}|]{\frac{4+2p}{5+2p}}^{N^2}\right) \tag{3.34}$$

with $p = \max \mathcal{A}$ and N the chosen constant in algorithm 1 for all finite alphabets $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ with $|\mathcal{A}| \geq 2$.

The computation of θ strongly follows [JP01]. We start with the fact that $\mathcal{L}_s$ is a nuclear operator. Using the notation in equation (3.16) the Ruelle operator can according to Theorem 3.1.8 be written as

$$\mathcal{L}_s v(z) = \sum_{n=0}^{\infty} \sum_{a \in \mathcal{A}} \left(\lambda_{a,n} l_n(v)\right) f_{a,n}(v). \tag{3.35}$$

We are interested in $\lambda_{a,n}$ for all $a \in \mathcal{A}$ and $n \in \mathbb{N}$. First we need to change indices such that $\lambda_{a,n}$ depends only on n.

Lemma 3.2.4. *Let $k = |\mathcal{A}|$. Let $\{a_i\}_{i=1}^{k} = \mathcal{A}$ be an ordering of $\mathcal{A}$ such that $a_1 < a_2 < \ldots < a_k$. The Ruelle operator can be written as*

$$\mathcal{L}_s v(z) = \sum_{n=1}^{\infty} \left(\tilde{\lambda}_n l_n(v)\right) \tilde{f}_n(v) \tag{3.36}$$

with $\tilde{\lambda}_{|\mathcal{A}|n+i} = \lambda_{a_i,n}$ and $\tilde{f}_{|\mathcal{A}|n+i} = f_{a_i,n}$ for all $n \in \mathbb{N}$ and $i \in \{1,\ldots,k\}$. Furthermore, $\tilde{\lambda}_n$ has the upper bound $\frac{25+10a_k}{4+2a_k} \sqrt[k]{\left(\frac{4+2a_k}{5+2a_k}\right)}^{n}$ for all $n \in \mathbb{N}$.

Proof. The first statement is a simple change of indices. Thus we will only prove the second part.

Let $i \in \{1,\ldots,k\}$ and $n \in \mathbb{N}$ be arbitrary. In Theorem 3.1.8 we have proven that $\lambda_{a_i,n} \leq 5\left(\frac{4+2a_k}{5+2a_k}\right)^n$ holds. This gives us

$$\tilde{\lambda}_{|\mathcal{A}|n+i} \leq 5\left(\frac{4+2a_k}{5+2a_k}\right)^n = \frac{25+10a_k}{4+2a_k} \sqrt[k]{\left(\frac{4+2a_k}{5+2a_k}\right)}^{|\mathcal{A}|(n+1)}. \tag{3.37}$$

From here the proof follows easily after a change of indices. ☐

In algorithm 1 we approximate $\det(I - z\mathcal{L}_s)$ by a Taylor series of $N+1$ terms. We then compute the zero is this Taylor series. The rate of convergence of algorithm 1 is the rate of decay of the coefficients in the Taylor series. To compute this rate we need another representation of the Taylor series which is mentioned in [JP01] and proven by Grothendieck [Gro95] and Fried [Fri86].

Lemma 3.2.5. *We can write* $\det(I - z\mathcal{L}_s) = 1 + \sum_{n=1}^{\infty} d_n(s) z^n$ *for all* $s \in [0,1]$ *and* $z \in D$ *with*

$$d_n(s) = (-1)^n \sum_{i_1 < \ldots < i_n} \left(\prod_{m=1}^{n} \tilde{\lambda}_{i_m} \right) \det \left(l_{i_j} \left(\tilde{f}_{i_k} \right) \right)_{j,k=1}^{n} \tag{3.38}$$

where l_{i_j} *are the functionals,* $\tilde{f}_{i_k}$ *are the functions and* $\tilde{\lambda}_{i_m}$ *are the constants defined in Lemma 3.2.4.*

We can compute the decay of the coefficients of the Taylor series of the Ruelle operator with more ease now. To estimate the determinant we will use Hadamard's theorem [Had93].

Theorem 3.2.6 (Hadamard (1893))**.** *Let* A *be an* $n \times n$*-matrix with* $\|a_{i,j}\| \leq 1$ *for all* $i, j \in \{1, \ldots, n\}$*. Then* $|\det(A)| \leq \sqrt{n^n}$*.*

Hadamard's theorem can be applied here since the functionals l_{i_j} and functions f_{a_k,i_k} are normalized. We need one more lemma called equation (3.7) in [JP01] to estimate the upper bound for θ.

Lemma 3.2.7 (Jenkinson, Pollicott (2001))**.** *Let* $\gamma \in (0,1)$ *and* $n \in \mathbb{N}$*. Then the following equation is true:*

$$\sum_{k_1 < \ldots < k_n} \gamma^{\sum_{i=1}^{n} k_i} = \frac{\sqrt{\gamma^{n(n+1)}}}{\prod_{r=1}^{n} (1 - \gamma^r)}. \tag{3.39}$$

Now we can generalize theorem 1(b) in [JP01].

Theorem 3.2.8. *Let* $N \in \mathbb{N}_{\geq 1}$ *be an number of terms of the Taylor approximation of* $\det(I - \mathcal{L}_s)$*. Let* s_N *be the returned value of algorithm 1. Then* s_N *converges to* $\dim_H(E_{\mathcal{A}})$ *as* N *grows to infinity with rate*

$$\mathcal{O}\left(\sqrt[2|\mathcal{A}|]{\frac{4+2p}{5+2p}}^{N^2} \right)$$

with $p = \max \mathcal{A}$*.*

Proof. Let $n \in \{1, \ldots, N\}$ and $s \in [0,1]$ be arbitrary. Let $p = \max \mathcal{A}$. Consider $d_n(s)$ as defined in Lemma 3.2.5. Let $\{i_m\}_{m=1}^{n} \subset \mathbb{N}$ be a strictly increasing sequence and let $\{a_m\}_{m=1}^{n} \subset \mathbb{N}$ be an increasing sequence. The proof of Theorem 3.1.8 gives us $\lambda_{a_m,i_m} \leq 5 \left(\frac{4+2p}{5+2p} \right)^{i_m}$ for all $m \in \{1, \ldots, n\}$. Now we can estimate $|d_n(s)|$ for all $s \in (0,1)$:

$$|d_n(s)| = \left| (-1)^n \sum_{i_1 < \ldots < i_n} \left(\prod_{m=1}^{n} \tilde{\lambda}_{i_m} \right) \det \left(l_{i_j} \left(\tilde{f}_{i_k} \right) \right)_{j,k=1}^{n} \right| \tag{3.40}$$

$$\leq \left(\frac{25+10p}{4+2p} \right)^n \left| \sum_{i_1 < \ldots < i_n} \sqrt[|\mathcal{A}|]{\left(\frac{4+2p}{5+2p} \right)}^{\sum_{m=1}^{n} i_m} \right| \sqrt{n^n} \tag{3.41}$$

$$= \left(\frac{25+10p}{4+2p} \right)^n \sqrt{n^n} \left(\frac{4+2p}{5+2p} \right)^{\frac{n(n+1)}{2|\mathcal{A}|}} \prod_{r=1}^{n} \left(1 - \left(\frac{4+2p}{5+2p} \right)^{\frac{r}{|\mathcal{A}|}} \right)^{-1} \tag{3.42}$$

$\mathcal{A}$	$\dim_H(E_\mathcal{A})$
$\{1,2\}$	$\mathbf{0,5312805}$
$\{1,2,3\}$	$\mathbf{0,7056}579$
$\{1,\dots,4\}$	$\mathbf{0,788}9709$
$\{1,\dots,5\}$	$\mathbf{0,836}8225$
$\{1,\dots,6\}$	$\mathbf{0,867}6452$
$\{1,\dots,7\}$	$\mathbf{0,88}89465$
$\{1,\dots,8\}$	$\mathbf{0,904}5715$
$\{1,\dots,9\}$	$\mathbf{0,916}4123$
$\{1,\dots,10\}$	$\mathbf{0,925}7507$

Table 3.1: Table with alphabet $\mathcal{A}$ and the found approximation $\dim_H(E_\mathcal{A})$ with $N = 10$, where the digits that are proven to be correct by [Jen04] are made bold.

$N\backslash\mathcal{A}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3,4\}$	$\{1,2,3,4,5\}$
1	0.332823034376	0.547484818846	0.648454617709	0.705879461020
2	0.549348313361	0.719505663961	0.801096122712	0.848104428500
3	0.532007087022	0.706163752824	0.789369124919	0.837214987725
4	0.531269412488	0.705653991550	0.788939978927	0.836824480444
5	0.531280446798	0.705660875887	0.788945529610	0.836829420179
6	0.531280506402	0.705660905689	0.788945559412	0.836829442531
7	0.531280506402	0.705660905689	0.788945559412	0.836829442531

Table 3.2: Table with various alphabets $\mathcal{A}$ on the top row and N from algorithm 1 first column. The entries are the returned s_N from algorithm 1.

In the last equality sign Lemma 3.2.7 is used. The above estimation can be bounded to

$$d_n(s) = \mathcal{O}\left(\sqrt[2|\mathcal{A}|]{\frac{4+2p}{5+2p}}^{n^2}\right). \tag{3.43}$$

Theorem 3.2.2 proved that s_N converges to $\dim_H(E_\mathcal{A})$ as N goes to infinity. Thus the terms in the Taylor polynomial of $\det(I - z\mathcal{L}_s)$ decay with rate $d_N(s)$ where $N+1$ is the number of terms in the truncated Taylor polynomial, which proves the theorem. □

3.3 Computations

In the appendix chapter A there is a Python code for algorithms 1 and 2. Two experiments have been run whose results can be seen in tables 3.1 and 3.2. Table 3.1 shows $\dim_H(E_\mathcal{A})$ for various $\mathcal{A}$ with $N = 10$ used with correctness. Table 3.2 showcases the convergence rate of s_N to $\dim_H(E_\mathcal{A})$. All tested $\mathcal{A}$ in table 3.2 have the first three decimals correct for $N = 4$ and higher. The approximation of $\dim_H\left(E_{\{1,2\}}\right)$ is accurate up to six digits for $N = 5$. This demonstrates that Jenkinson-Pollicott's algorithm indeed is a very fast algorithm for computing the Hausdorff dimension of $E_\mathcal{A}$.

Chapter 4

Zaremba's conjecture

One application of Jenkinson-Pollicott's algorithm is Bourgain-Kontorovich's attempt [BK14] to prove Zaremba's conjecture [Zar72] as mentioned in the introduction.

Conjecture 4.0.1 (Zaremba's conjecture (1972))**.** *Let $A \in \mathbb{N}_{\geq 1}$ be a positive integer and let $\mathcal{A} = \{1, \ldots, A\}$ be a finite alphabet. Let*

$$\mathscr{R}_A = \left\{ x \in [0,1] \,\middle|\, \exists \{a_i\}_{i=1}^{k} \subset \mathcal{A} : x = \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}} \right\} \tag{4.1}$$

be the set of finite continued fractions with coefficients in $\mathcal{A}$. Let

$$\mathscr{D}_A = \left\{ d \in \mathbb{N} \,\middle|\, \exists b \in \mathbb{N} : \frac{b}{d} \in \mathscr{R}_A \wedge \gcd(b,d) = 1 \right\} \tag{4.2}$$

be the set of the denominators of the elements in $\mathscr{R}_A$. Then there is an $Z \in \mathbb{N}$ such that $\mathscr{D}_Z = \mathbb{N}$ holds.

Remark 4.0.2. Zaremba suggested that his conjecture is true for $A = 5$. Bourgain and Kontorovich [BK14] proved Theorem 4.1.1 which was a significant progress. In 2015 Huang [Hua15] proves that Bourgain's and Kontorovich's Theorem 4.1.2 is true for $A = 5$.

Kontorovich [Kon13] explained very well how Zaremba came up with his conjecture and what the consequences of the conjecture are. In this chapter we will take a look at the article on Zaremba's conjecture by Bourgain and Kontorovich [Zar72], in particular what their main results are. We will then focus on how they used $\dim_H(E_{\mathcal{A}}) \geq c > 0$ with $c \in (0,1)$ for a finite alphabet $\mathcal{A} \subset \mathbb{N}_{\geq 1}$ to prove $\lim_{N\to\infty} \frac{1}{N} |\mathscr{D}_A \cap [1,N]| = 1$ for an $A \in \mathbb{N}_{>1}$.

4.1 Overview

We first start with some definitions. Let $A \in \mathbb{N}_{\geq 2}$ be an integer and consider for this chapter the alphabet $\mathcal{A} = \{1, \ldots, A\}$. Define $\mathscr{D}_A^q := \{d \mod q \,|\, d \in \mathscr{D}_A\}$

for all $q \in \mathbb{N}$. An integer $d \in \mathbb{N}$ is *admissible* for $\mathcal{A}$ if $d \in \mathscr{D}_{\mathcal{A}}^q$ holds for all $q \in \mathbb{N}_{\geq 2}$, thus $d \in \mathbb{N}$ is admissible is there are no $q \in \mathbb{N}_{\geq 2}$ with $d + kq \notin \mathscr{D}_{\mathcal{A}}$ for all $k \in \mathbb{Z}$. Let

$$\mathscr{U}_A = \{d \in \mathbb{Z} \,|\, \forall q \in \mathbb{N}_{\geq 2} : d \in \mathscr{D}_A^q\} \tag{4.3}$$

be the set of all admissible numbers for $\mathcal{A}$. Then Bourgain and Kontorovich proved the following theorem in [BK14].

Theorem 4.1.1 (Bourgain, Kontorovich (2014))**.** *Let $A \in \mathbb{N}_{\geq 1}$ be a positive integer and $\mathcal{A} = \{1, \dots, A\}$ be an alphabet. If $\dim(E_{\mathcal{A}}) > \frac{307}{312}$, then the set $\mathscr{D}_A$ contains almost every admissible integer. This means that there is a constant $c \in \mathbb{R}_{>0}$, depending only on $\mathcal{A}$, such that*

$$\frac{\left|\mathscr{D}_A \cap \left[\frac{N}{2}, N\right]\right|}{\left|\mathscr{U}_A \cap \left[\frac{N}{2}, N\right]\right|} = 1 + \mathcal{O}\left(\exp\left(-c\sqrt{\log N}\right)\right) \tag{4.4}$$

holds as $N \to \infty$.

First take a look at $\mathscr{U}_A$. In [BK14] it is proven that $\mathscr{U}_A = \mathbb{Z}$ holds for all $A \in \mathbb{N}_{\geq 2}$, thus every integer is admissible for $\mathcal{A}$. Furthermore, Jenkinson [Jen04] proved that $\dim_H\left(E_{\mathcal{A}\cup\{j\}}\right) > \dim_H(E_{\mathcal{A}})$ holds for every $j \in \mathbb{N}_{\geq 1} \backslash \mathcal{A}$ with upper bound $\dim_H\left(E_{\mathbb{N}_{\geq 1}}\right) = \dim_H([0,1]) = 1$. Thus there is an $A \in \mathbb{N}$ such that $\dim_H\left(E_{\{1,\dots,A\}}\right) \geq \frac{307}{312}$ holds. These two statements give rise to the following theorem, which is the main result of [BK14].

Theorem 4.1.2 (Bourgain, Kontorovich (2014))**.** *There exists an $A \in \mathbb{N}_{\geq 2}$ such that*

$$\lim_{N\to\infty} \frac{1}{N} |\mathscr{D}_A \cap [1, N]| = 1. \tag{4.5}$$

Thus $\mathscr{D}_A$ has almost every denominator of $\mathscr{R}_A$ for a then unknown $A \in \mathbb{N}_{\geq 2}$. Huang [Hua15] proved $A = 5$ suffices. Note that this is very close to proving $\mathscr{D}_5 = \mathbb{N}$ which has Zaremba's conjecture as corollary.

4.2 Use of Hausdorff dimension

In the last section of this thesis we will look at the use of Hausdorff dimension in the proof of Theorem 4.1.1 given by [BK14]. First we need to reformulate Conjecture 4.0.1. Let $b, d \in \mathbb{N}_{\geq 1}$ with $\gcd(b, d) = 1$ and let $\{a_i\}_{i=1}^k \subset \mathcal{A}$ be a sequence such that $\frac{b}{d} = [a_1, \dots, a_k]$ holds. Then, according to [BK14, Kon13],

$$\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_k \end{pmatrix} = \begin{pmatrix} * & b \\ * & d \end{pmatrix} \tag{4.6}$$

holds where the asterisks denote unimportant numbers. Let Γ_A be the semi-group generated by $\left\{\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} \middle| a \in \{1, \dots, A\}\right\}$. Let

$$O_A := \Gamma_A e_2 = \left\{\begin{pmatrix} b \\ d \end{pmatrix} \in \mathbb{N}_{\geq 1}^2 \,\middle|\, \exists a, c \in \mathbb{Z} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_A\right\} \tag{4.7}$$

be the orbit that isolates b and d from Γ_A for the quotients $\frac{b}{d} \in \mathscr{R}_A$ with $\gcd(b, d) = 1$. Then $\mathscr{D}_A$ equals $\mathscr{D}_A = \langle e_2, O_A\rangle = \langle e_2, \Gamma_A e_2\rangle$, thus Zaremba's conjecture is equivalent to the following conjecture:

Conjecture 4.2.1. *There is an $A \in \mathbb{N}_{\geq 2}$ such that $\mathscr{D}_A = \langle e_2, O_A \rangle = \mathbb{N}_{\geq 1}$ holds.*

Let $N \in \mathbb{R}_{>1}$. Define

$$\mathscr{R}_A(N) := \left\{ \frac{b}{d} \in \mathscr{R}_A \,\middle|\, \gcd(b,d) = 1 \wedge 1 \leq b < d < N \right\}. \tag{4.8}$$

Then Hensley [Hen89] proved that $|\mathscr{R}_A(N)| = \Theta\left(N^{2\dim_H(E_{\mathcal{A}})}\right)$ where $f = \Theta(g)$ means $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$. Let Ω_N be the set $\mathscr{R}_A(N)$ but written in terms of the reformulation:

$$\Omega_N \subset \{\gamma \in \Gamma \,|\, |\gamma| < N\}. \tag{4.9}$$

Then in [Kon13] it is noted that $|\Omega_N| = \Theta\left(N^{2\dim_H(E_{\mathcal{A}})}\right)$ holds as well. Now we want to count how often an integer appears as denominator in the orbit of Γe_2. Define

$$\mathcal{R}_N(n) := \sum_{\gamma \in \Omega_N} \mathbb{1}_{\{n = \langle e_2, \gamma e_2 \rangle\}}(\gamma). \tag{4.10}$$

Now we want to see how much $\mathcal{R}_N(n)$ grows as N tends to ∞. This can be done by looking at the Fourier transform of $\mathcal{R}_N(n)$:

$$\mathcal{S}_N(\theta) := \sum_{n \in \mathbb{Z}} \mathcal{R}_N(n) e^{2n\theta\pi i} = \sum_{\gamma \in \Omega_N} e^{2\langle e_2, \gamma e_2 \rangle \theta \pi i}. \tag{4.11}$$

Then $\mathcal{R}_N$ can be recovered by $\mathcal{R}_N(n) = \int_{\mathbb{R}\backslash\mathbb{Z}} \mathcal{S}_N(\theta) e^{-2n\theta\pi i} d\theta$. Hardy and Littlewood considered looking at a certain arc $\mathfrak{M} \subset [0,1)$ specified in [BK14] such that $\mathcal{R}_N(n)$ splits up in $\mathcal{M}_N(n)$ and $\mathcal{E}_N(n)$ where

$$\mathcal{M}_N(n) = \int_{\mathfrak{M}} \mathcal{S}_N(\theta) e^{-2n\theta\pi i} d\theta \tag{4.12}$$

is the main term and

$$\mathcal{E}_N(n) = \int_{[0,1)\backslash\mathfrak{M}} \mathcal{S}_N(\theta) e^{-2n\theta\pi i} d\theta \tag{4.13}$$

is the error. We will from now consider $\mathcal{E}_N(n)$. Bourgain and Kontorovich proved the following theorem in [BK14].

Theorem 4.2.2 (Bourgain, Kontorovich (2014))**.** *If $\dim_H(E_{\mathcal{A}}) > \frac{307}{312}$ holds, there is a $c \in \mathbb{R}_{>0}$ such that*

$$\sum_{n \in \mathbb{Z}} |\mathcal{E}_N(n)|^2 = \mathcal{O}\left(\frac{|\Omega_N|^2}{N} e^{-c\sqrt{\log N}} \right). \tag{4.14}$$

Bourgain and Kontorovich proved in [BK14] a similar bound for the main term.

Theorem 4.2.3 (Bourgain, Kontorovich (2014))**.** *Let $n \in \left[\frac{1}{20}N, \frac{1}{10}N\right) \cap \mathbb{Z}$. Then*

$$\mathcal{M}_N(n) = \mathcal{O}\left(\frac{N}{|\Omega_N|} \log\log N \right). \tag{4.15}$$

Bourgain and Kontorovich used these theorems to prove their Theorem 4.1.1, from which the main result of their paper, Theorem 4.1.2, followed.

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Appendix A

Jenkinson-Pollicott's algorithm code

```
import math
import numpy as np
import smtplib

def maak_rijen(A,N):
    #print("Voer stap 1 uit")
    stappen = 0
    #output = open('rijtjes.txt', 'r+')
    #output.truncate(0)

    for i in range(1,N+1):
        stappen += len(A) ** i
    teller = 0. #Houdt voortgang van script bij

    rijtjes = []
    for i in A: #Maak alle rijtjes van lengte 1
        rijtjes.append([i])
        #output.write('%s\n' % (i))
        teller += 1

    a = 0; b = 0
    for n in range(2,N+1):
        a = b; b += len(A) ** (n-1)
        #Itereer over de juiste waarden
        for i in A: #Nieuw rijtje lengte n begint met i
            for k in range(a,b): #Rijtje lengte n-1 na i
                hulp = [i]
                rijtje = rijtjes[k]
                #print("%s '%" % (teller/stappen*100))
                for j in rijtje:
                    hulp.append(j)
                rijtjes.append(hulp)
```

```
                teller += 1
                """for j in range(len(hulp)):
                    output.write('%s ' % (hulp[j]))
                output.write('\n')
    output.close()"""
    return rijtjes

def kettingbreuk(rij):
        #print("Voer stap 2 uit")
        breuk = 0.
        hulp = 1. #Nodig om while te starten
        while abs(hulp - breuk) > 10 ** -8:
            hulp = breuk
            for i in reversed(rij):
                breuk = 1./(i+breuk)
        return breuk

def gewicht(rij):
    #print("Voer stap 3 uit")
    resultaat = [];
    breuk = [kettingbreuk(rij)];
    totaal = 0
    resultaat.append(np.log(breuk[0]))
    if len(rij) > 1:
        for r in range(1,len(rij)):
            nieuw = 1./breuk[r-1]
            oud = math.floor(1./breuk[r-1])
            breuk.append(nieuw - oud)
            resultaat.append(math.log(breuk[r]))
    for i in range(0,len(rij)): totaal += resultaat[i]
    return math.exp(totaal)

def f_4(rij,s):
    #print("Voer stap 4 uit")
    teller = gewicht(rij) ** (2*s)
    noemer = (1-(-1)**len(rij)*gewicht(rij) ** 2)
    return  teller / noemer

def c(n,s):
    #print("Bekereken de c_n(s) zoals in stap 5")
    b = 0; a = 0
    for i in range(1,n+1):
        a = b; b += len(A)**(i)
    totaal = 0.
    for rij in range(a,b):
        totaal += f_4(rijen[rij],s)
    totaal /= n
    return totaal

def partitie(number):
```

```
    answer = []
    answer.append([number])
    for x in range(1, number):
        for y in partitie(number - x):
            answer.append([x] + y)
    return answer

def d(m,s):
    #print("Voer stap 6 uit met d_m(s)")
    resultaat = 0
    lijst = partitie(m)
    for rij in lijst:
        som = 0
        for l in rij:
            som += np.log(c(l,s))
        teller = np.exp(som)*(-1) ** len(rij)
        noemer = math.factorial(len(rij))
        resultaat += teller / noemer
    return resultaat

def Delta(s):
    #print("Voer stap 7 uit")
    totaal = 1
    for m in range(1,N+1):
        totaal += d(m,s)
    return totaal

def Delta_solver(start):
    #print("Stap 8")
    a = 0.; b = 1.
    Teller = 0
    while abs(a-b) > 10 ** -8:
        test = np.float64(Delta(start))
        if test < 0:
            a = start
            start = np.float64(a + b)/2
        elif test > 0:
            b = start
            start = np.float64(a + b)/2
        else: return test
        Teller += 1
        #print("Testwaarde is %s" % (start))
        #print("Iteratie %s" % (Teller))
    return start

A = [1]
#rijen = maak_rijen(A,N)
#print(Delta_solver(0.5))
""" Bereken dim_H(E_A) bij verschillende A en nauwkeurigheid."""
```

```
output = open("nauwkeurigheid.txt","w")
output.truncate(0)
for i in range(2,10):
    A.append(i)
    for N in range(1,8):
        rijen = maak_rijen(A,N)
        output.write("%s, " % (Delta_solver(0.5)))
        print("Nauwkeurigheid %s van %s is weggeschreven"
                % (N,A))
    print("\n")
output.close()

""" Uncomment bij Hausdorff-dimensie berekening
output = open("dimensies.txt","w")
output.truncate(0)
for i in range(2,11):
    A.append(i)
    rijen = maak_rijen(A,N)
    output.write("%s\n" % (Delta_solver(0.5)))
    print("Dimensie van %s is weggeschreven" % (A))
output.close()
"""
```