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Integrally closed monomial ideals in dimension two and powers of ideals

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Filosofie licentiatavhandling

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1 Introduction

In the appendix of [4] and in a paper by C. Huneke in [2] one can find two basic theorems on integrally closed ideals in two-dimensional regular local rings. Firstly that the product of integrally closed ideals is again integrally closed. Secondly that every integrally closed ideal factors uniquely into a product of simple integrally closed ideals. In this thesis we present an approach to the case of monomial ideals in $k[x, y]$. In Section 3 we determine all integrally closed monomial ideals and show that there is a one-to-one correspondence with ascending chains of positive rational numbers. Section 4 describes powers of certain ideals in integral domains, which in the case of two-dimensional polynomial rings has some connection to integrally closed domains.

2 Background

We begin this section by stating some wellknown properties of integral closure of an ideal. The reader may consult [1]. In Subsections 2.2 and 2.3 we continue with the special case of monomial ideals; we have used some ideas from [3].

2.1 Integral closure of ideals

An element $x \in R$ is said to be *integral* over I, if x satisfies an equation

$$x^d + a_1 x^{d-1} + \cdots + a_{d-1} x + a_d = 0 \tag{2.1}$$

where $a_j \in I^j$. The *integral closure* of I, denoted by $\bar{I}$, is defined as the set of all elements in R which are integral over I. Integral closure of an ideal can also be defined using the well known definition of integral closure of a ring.

Proposition 2.1. *Let $x \in R$, $I \subset R$ and $R[It] = R \oplus It \oplus I^2t^2 \oplus ... \subset R[t]$ be the Rees ring with respect to I. Then $x \in \bar{I}$ if and only if $xt \in \overline{R[It]}$.*

Proof. Let $x \in \bar{I}$ and consider an equation of integral dependence (1.1) of x over I. Multiplying it by t^d, the resulting equation is

$$(xt)^d + (a_1 t)(xt)^{d-1} + \cdots + (a_{d-1} t^{d-1})(xt) + (a_d t^d) = 0, \tag{2.2}$$

where $(a_j t^j) \in I^j t^j \subset R[It]$. That is, $xt \in R[t]$ is integral over $\overline{R[It]}$.

On the other hand, if $xt \in \overline{R[It]}$, then xt satisfies an equation $(xt)^d + r_1(xt)^{d-1} + \cdots + r_{d-1}(xt) + r_d = 0$ where $r_j \in R[It]$. Now $R[t]$ is graded in the usual way and $R[It]$ is a graded subring of it, so the equation of integral dependence of xt can be split into its homogeneous parts. Taking the part of degree d we get a homogeneous equation that looks like the one in (1.2). Cancelling t^d we get $x \in \bar{I}$. $\square$

Corollary 2.2. *Let $I \in R$, then the integral closure $\bar{I}$ is an ideal. Moreover, $\bar{I}$ has the same radical as I.*

Proof. Let $x, y \in \bar{I}$ and $r \in R$. Multiplying (1.1) with r^d we have $(rx)^d + (a_1 r)(rx)^{d-1} + \cdots + (a_{d-1} r^{d-1})(xr) + (a_d r^d) = 0$, an equation of integral dependence of rx over I. Thus $rx \in \bar{I}$. Since the set $\overline{R[It]}$ forms a subring of $R[t]$ we also have $x, y \in \bar{I} \Leftrightarrow xt, yt \in \overline{R[It]} \Rightarrow xt + yt = (x+y)t \in \overline{R[It]} \Leftrightarrow x + y \in \bar{I}$. This makes $\bar{I}$ to an ideal.

The inclusion $\operatorname{rad} I \subseteq \operatorname{rad} \bar{I}$ follows from $I \subseteq \bar{I}$. Next, let $x \in \operatorname{rad} \bar{I}$, then $x^l \in \bar{I}$ for some positive integer l, say $(x^l)^d + a_1 (x^l)^{d-1} + \cdots + a_{d-1}(x^l) + a_d = 0$. Clearly, $x^{ld} \in I$ and hence $x \in \operatorname{rad} I$. Thus, $\operatorname{rad} I = \operatorname{rad} \bar{I}$. ☐

Saying that J is integral over I means simply that each element belonging to J is integral over I.

Corollary 2.3. *Let I, J, $K \subset R$ be ideals such that J is integral over I and K is integral over J. Then K is integral over I.*

Proof. If J is integral over I, then $Jt \subset R[Jt]$ is integral over $R[It]$. The ring R is obviously integral over $R[It]$. Thus, the ring $R[Jt]$, which is generated by R and Jt, is integral over $R[It]$. It follows also that the ring $R[Kt]$ is integral over $R[Jt]$, and therefore integral over $R[It]$. That is, K is integral over I. ☐

2.2 Integral closure of monomial ideals

Throughout the paper it will be tacitly understood that $I = \langle m_i \rangle \subset k[X] = k[x_1, \ldots, x_n]$ means an ideal generated by monomials, a monomial ideal.

Definition 2.4. A *power product* in $k[X]$ is an element $X^{\mathrm{a}} = x_1^{a_1} \cdots x_n^{a_n}$. All the power products in a polynomial ring form a basis for it over k. Thus, every polynomial $p \in k[X]$ can be written as $p = \sum c_i X^{\mathrm{a}_i}$ where all the X^{a_i} are different; if $c_i \neq 0$ then we say that $c_i X^{\mathrm{a}_i}$ is a *monomial in* p (or that X^{a_i} is a *power product in* p).

Remark 2.5. Any monomial ideal I in $k[X]$, where k is a field, is of course an ideal generated by power products. Let now p be a polynomial belonging to a monomial ideal $\langle X^{\mathrm{a}_1}, \ldots, X^{\mathrm{a}_q} \rangle$, then $p = r_1 X^{\mathrm{a}_1} + \cdots + r_q X^{\mathrm{a}_q}$, where each $r_i \in k[X]$. It is easy to see that every power product in p is a monomial multiplied by some $X^{\mathrm{a_i}}$. Thus, every monomial in p must belong to I.

We continue with a lemma necessary for showing one basic property of integral closure of monomial ideals.

Lemma 2.6. *Let $R = k[X]$, $I \subset R$ an ideal and $m_1, \ldots, m_q$, where $q \geq 2$, a set of different power products such that*

$$\begin{aligned} m_1^{s_1} &\in I^{s_1 - r_1} \langle m_2, m_3, \ldots, m_q \rangle^{r_1} \\ m_2^{s_2} &\in I^{s_2 - r_2} \langle m_1, m_3, \ldots, m_q \rangle^{r_2} \\ &\vdots \\ m_q^{s_q} &\in I^{s_q - r_q} \langle m_1, m_2, \ldots, m_{q-1} \rangle^{r_q} \end{aligned} \tag{2.3}$$

for some integers $0 \le r_i \le s_i$, $s_i > 0$. Then $m_i^{l_i} \in I^{l_i}$ for $1 \le i \le q$ and for some $l_i > 0$.

Proof. We use induction on q. Let $q = 2$. After raising m_1 to the power s_2 and using the condition on $m_2^{s_2}$ we get:

$$m_1^{s_1 s_2} \in I^{(s_1-r_1)s_2} m_2^{r_1 s_2} \subseteq I^{(s_1-r_1)s_2}(I^{s_2-r_2}\langle m_1\rangle^{r_2})^{r_1} = I^{s_1s_2-r_1r_2}\langle m_1^{r_1r_2}\rangle$$

$$m_1^{s_1s_2-r_1r_2} \in I^{s_1s_2-r_1r_2}.$$

Doing the same for m_2 shows that the lemma is valid for $q = 2$.

Let $q \ge 3$. Consider the first and last relation in (2.3). By "factoring out" the necessary monomial respectively they can be written as

$$m_1^{s_1} \in I^{s_1-r_1}\langle m_2, m_3, \ldots, m_{q-1}\rangle^{r_1-r_1'} m_q^{r_1'},$$
$$m_q^{s_q} \in I^{s_q-r_q}\langle m_2, m_3, \ldots, m_{q-1}\rangle^{r_q-r_q'} m_1^{r_q'}.$$

Then we consider $m_1^{s_1s_q}$ and rewrite it using the relation for $m_q^{s_q}$:

$$(m_1^{s_1})^{s_q} \in I^{s_1s_q-r_1s_q+r_1's_q-r_1'r_q}\langle m_2,\ldots,m_{q-1}\rangle^{r_1s_q-r_1's_q+r_1'r_q-r_1'r_q'} m_1^{r_1'r_q'}.$$

Hence, for some integers $0 \le R_1 \le S_1 \le s_1s_q$, where $S_1 > 0$, we have $m_1^{S_1} \in I^{S_1-R_1}\langle m_2,\ldots,m_{q-1}\rangle^{R_1}$.

Repeating the procedure for each pair m_i and m_q, where $2 \le i \le q-1$ we eliminate m_q from the relations for $m_1,\ldots,m_{q-1}$ and the induction hypothesis yields the result for these monomials. The choice of m_q to be eliminated from the relations is arbitrary, which finishes the proof. $\square$

Proposition 2.7. *Let $R = k[X]$ and $I \subset R$ be a monomial ideal. Then the integral closure $\bar{I}$ is also a monomial ideal.*

In fact, $\bar{I} = \langle m \mid m^l \in I^l$ for some $l > 0\rangle$.

Proof. Let $m \in \bar{I}$, i.e. $m^d + a_1m^{d-1} + \cdots + a_{d-1}m + a_d = 0,\ a_j \in I^j$. Then $m^d = n_l m^{d-l}$ for some $1 \le l \le d$ and a monomial $n_j \in I^j$ (cf. Remark 2.5). Thus, $m^l \in I^l$.

Next, let $p = m_1 + \cdots + m_q \in \bar{I}$, where $q \ge 2$ and $m_1,\ldots,m_q$ are the monomials in some p. Let

$$\begin{aligned}&(m_1+\cdots+m_q)^d + a_1(m_1+\cdots+m_q)^{d-1} + \cdots + a_{d-1}(m_1+\cdots+m_q) + a_d = \\ &= m_1^d + \cdots + m_q^d + \cdots + a_d = 0\end{aligned} \tag{2.4}$$

be an equation of integral dependence for p. From (2.4) we see that $m_1^d = n\prod_{i=1}^q m_i^{t_i} = m_1^{t_1}\cdot n\prod_{i\ne1} m_i^{t_i}$ where $d > t_1$ and where n is either a monomial belonging to $I^{d-\sum t_i}$ or $n \in k$. Thus $m_1^{d-t_1} = n\prod_{i\ne1} m_i^{t_i}$. In the same way we obtain similar relations for $m_2,\ldots,m_q$. The conditions for $m_1,\ldots,m_q$ in

Lemma 2.6 are then fulfilled and we have that $m_i^{l_i} \in I^{l_i}$ for some l_i, whence $m_i \in \bar{I}$. This proves that $\bar{I}$ is monomial.

We have at the same time proved the inclusion $\subseteq$ in the second part of our proposition. The other inclusion is clear and we are done. $\square$

2.3 Graphical representation

A monomial ideal I in n variables can be visualized graphically, by representing the set of exponents of power products in I as lattice points in $\mathbb{R}^n$. Such a representation will be essential for all the results we are going to present.

Definition 2.8. Let $X^{\mathrm{a}} = x_1^{a_1} \cdots x_n^{a_n}$ be a power product in $R = k[X]$, then we set $\log X^{\mathrm{a}} = \mathrm{a}$. Given a monomial ideal I we define the semigroup ideal

$$\log I = \{\log m \mid m \in I,\ m \text{ a power product}\}.$$

In this language Proposition 2.7 says that $\log \bar{I} = \{\mathrm{a} \in \mathbb{N}^n \mid \mathrm{a}l \in \log (I^l)$ for some $l > 0\}$. Further we get to a nice description of the integral closure, stated below. Notice that we do not need to use generators of I neither in the proposition nor in the proof.

First we recall the definition of a convex hull of a set S, which is $\mathrm{conv}(S) = \{\lambda_1\mathrm{a}_1 + \cdots + \lambda_q\mathrm{a}_q \,|\, \mathrm{a}_i \in S\}$, where all $\lambda_i \in \mathbb{R}_{\geq 0}$ and $\sum_{i=0}^{q} \lambda_i = 1$. If all $\lambda_i \in \mathbb{Q}_{\geq 0}$ then the convex hull is called *rational* and is denoted by $\mathrm{conv}_{\mathbb{Q}}$.

Proposition 2.9. *Let $I \subset k[X]$ be a monomial ideal. Then the integral closure $\bar{I}$ is generated by monomials whose exponents are lattice points in the rational convex hull of* $\log I$. *That is,*

$$\log \bar{I} = \mathrm{conv}_{\mathbb{Q}}(\log I) \cap \mathbb{N}^n.$$

Proof. Let $\mathrm{a} \in \mathrm{conv}_{\mathbb{Q}}(\log I) \cap \mathbb{N}^n$. Then $\mathrm{a} = \sum_{i=1}^{q} \lambda_i \mathrm{a}_i$, where all $\mathrm{a}_i \in \log I,\ \lambda_i \in \mathbb{Q}_{\geq 0}$ and $\sum_{i=1}^{q} \lambda_i = 1$. Since there is an integer $l > 0$ such that $l\lambda_i \in \mathbb{N}^n$ for all i, we obtain $(X^{\mathrm{a}})^l = (X^{\sum \lambda_i \mathrm{a}_i})^l = (X^{\mathrm{a}_1})^{l\lambda_1} \cdots (X^{\mathrm{a}_q})^{l\lambda_q} \in I^l$. Thus $X^{\mathrm{a}} \in \bar{I}$, that is, $\mathrm{a} \in \log \bar{I}$.

On the other hand, for any $\mathrm{b} \in \log \bar{I}$ there is an integer l such that $\mathrm{b}l = \mathrm{b}_1 + \cdots + \mathrm{b}_l$ where every b_i (not necessarily different) belongs to $\log I$ (cf. Proposition 2.7). Thus, $\mathrm{b} = \sum_{i=1}^{l} \frac{1}{l}\mathrm{b}_i$ and it follows that $\mathrm{b} \in \mathrm{conv}_{\mathbb{Q}}(\log I)$. $\square$

3 Integrally closed monomial ideals in two variables

In this section we will determine all integrally closed monomial ideals in the ring $k[x, y]$ using the graphical interpretation described previously. We will also show how they can be factorized into simple ones.

It follows directly from Proposition 2.9 that principal monomial ideals are integrally closed.

Lemma 3.1. *Let $J \subset k[x,y]$ be a non-principal monomial ideal and assume that $J = mI$ where m is a power product and I is a monomial ideal. Then J is integrally closed if and only if I is integrally closed.*

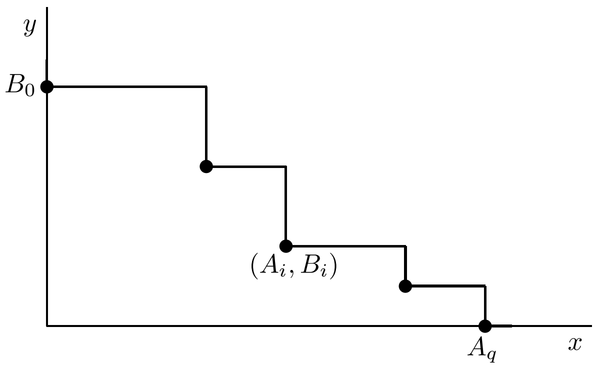
Proof. It is clear that $\operatorname{conv}_{\mathbb{Q}}(\log I) = \log(m) + \operatorname{conv}_{\mathbb{Q}}(\log J)$ which gives the result. $\square$

We recall that a monomial ideal $I \subset k[x,y]$ is $\langle x,y\rangle$-primary if and only if there are positive integers A and B such that x^A and y^B belong to I. If $mI = J$ is a non-principal monomial ideal and where m is the greatest common divisor for the generators of J, then I is $\langle x,y\rangle$-primary. Thus, we can limit our subject of interest to $\langle x,y\rangle$-primary monomial ideals when we study integrally closed monomial ideals in $k[x,y]$.

3.1 Necessary conditions for integral closedness

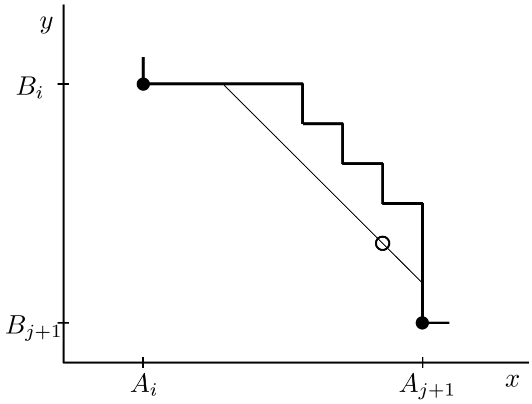
We can always write an ideal as $I = \langle y^{B_0}, \ldots, x^{A_i}y^{B_i}, \ldots, x^{A_q}\rangle$, where $A_i < A_{i+1}$ and $B_i > B_{i+1}$. These generators form a minimal generating set for I. Henceforth it will be understood that the $\langle x,y\rangle$-primary monomial ideals we are considering are always written in such a way.

The semigroup ideal $\log I$ is then $\{(0,B_0),\ldots,(A_i,B_i),\ldots,(A_q,0)\}+\mathbb{N}^2$ and can be graphically interpreted as the lattice points on and above the staircase depicted below. A pair of consecutive generators $x^{A_i}y^{B_i}$ and $x^{A_{i+1}}y^{B_{i+1}}$ will make a step having *breadth* $A_{i+1} - A_i$ and *height* $B_i - B_{i+1}$.



We know that the lattice points in $\operatorname{conv}(\{(A_i,B_i)_{i=0}^q\})$ generate $\bar{I}$. In order to find some necessary condition on a monomial ideal to be integrally closed we look at the convex hull of two consecutive exponents, $\operatorname{conv}(\{(A_i,B_i), (A_{i+1},B_{i+1})\} + \mathbb{N}^2)$, particularly this area contains the triangle with vertices (A_i,B_i), (A_{i+1},B_i) and (A_{i+1},B_{i+1}). In an integrally closed ideal there must not be any lattice points in this triangle. It is obvious that this is the case if and only if either $A_{i+1} - A_i = 1$ or $B_i - B_{i+1} = 1$. Thus, in an integrally closed monomial ideal every generator (except for the last one) is followed by a generator that has either a power of x which is increased by one or a power of y which is decreased by one.

Assume that the condition above is fulfilled for each pair of consecutive generators. Assume further that $A_{i+1} - A_i \geq 2$ and $B_j - B_{j+1} \geq 2$ for some $i \leq j$, where i is the greatest index and j is the smallest index such that the situation occurs.



The diagonal line in the figure begins at the point $(A_{i+1}-2, B_i)$ and ends at $(A_{j+1}, B_j - 2)$. Clearly, the lattice points in the area above and to the right of the line belong to the integral closure, particularly the point $(A_{j+1}-1, B_j-1)$. Thus, the corresponding power product $x^{A_{j+1}-1}y^{B_j-1}$ will always appear in $\bar{I}$, which means that under these conditions the ideal cannot be integrally closed.

We have shown that in an integrally closed ideal we have always the following situation: if the power of x increases by at least two somewhere among the generators, then the power of y must decrease only by one among the following generators. (If some step has breadth at least two, then the following steps must have height one only.) Thus, a necessary condition for a monomial ideal to be integrally closed is that the generating set consists of two parts, where the powers of x increase by one in the first part and the powers of y decrease by one in the second part. We formulate our reasoning algebraically in a proposition.

Proposition 3.2. *Let $I \subset k[x,y]$ be an integrally closed $\langle x, y\rangle$-primary monomial ideal. Then this ideal can be written as*

$$I = \langle y^{s+B_0}, \ldots, x^i y^{s+B_i}, \ldots, x^r y^s, \ldots, x^{r+A_j}y^{s-j}, \ldots, x^{r+A_s}\rangle,$$

where $B_i > B_{i+1}$ and $A_i < A_{i+1}$, or as

$$I = \left\langle y^s \langle x^i y^{B_i}\rangle_{i=0}^{r}, x^r \langle x^{A_j}, y^{s-j}\rangle_{j=0}^{s} \right\rangle,$$

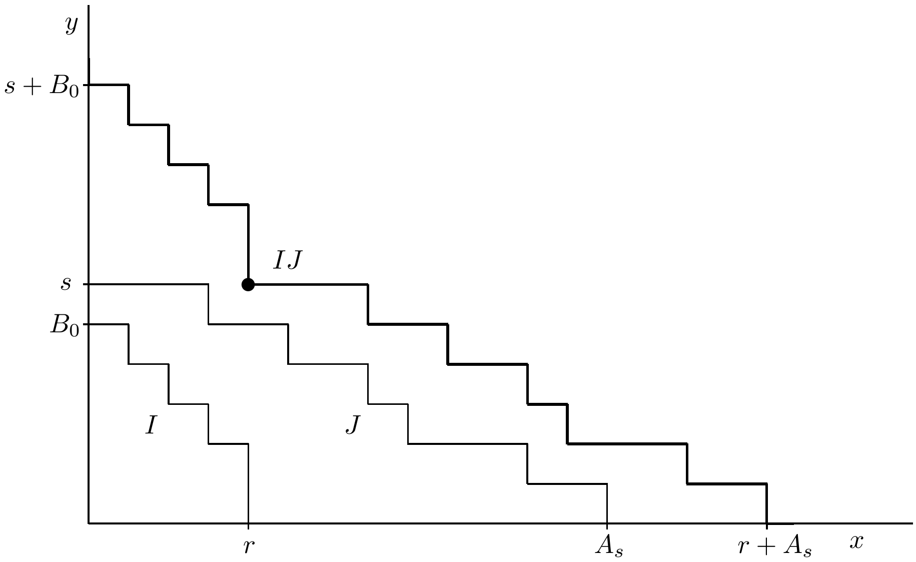
where in addition $B_r = 0 = A_0$.

Actually, ideals of the form described above can be factorized in a very convenient way. Notice that not all such ideals are integrally closed. Hence, the factorization proposition that follows is valid more generally and not only for integrally closed ideals.

Proposition 3.3. *Let* $I = \langle x^i y^{B_i} \rangle_{0 \le i \le r}$ *where* $B_i > B_{i+1}$ *and* $B_r = 0$*, and* $J = \langle x^{A_j} y^{s-j} \rangle_{0 \le j \le s}$ *where* $A_i < A_{i+1}$ *and* $A_0 = 0$*. Then* $IJ = y^s I + x^r J$*.*

Moreover, the product IJ *is integrally closed if and only if* I *and* J *are both integrally closed.*

Graphically the ideal I is a staircase where each step has breadth one, while all the steps in J have height one. The statement of the proposition is that the product of I and J is the staircase I followed by J.



Proof. The part $y^s I + x^r J \subseteq IJ$ is clear.

Next we will show that any power product $x^{i+A_j} y^{B_i+s-j}$, that generates IJ, must belong to either $y^s I$ or $x^r J$.

In case $i + j \le r$ we have the following inequalities (easily derived from the conditions on A_j and B_i):

$$\left\{ \begin{array}{ll} A_j & \ge j \\ B_i - j & \ge B_{i+j} \end{array} \right. \text{ or: } \left\{ \begin{array}{ll} i + A_j & \ge i + j \\ B_i + s - j & \ge s + B_{i+j} \end{array} \right. .$$

That is, $x^{i+A_j} y^{B_i+s-j} \in y^s I$.

On the other hand, if $i + j \ge r$ then we have:

$$\left\{ \begin{array}{ll} A_j - (r - i) & \ge A_{j-(r-i)} \\ B_i & \ge r - i \end{array} \right. \text{ or: } \left\{ \begin{array}{ll} i + A_j & \ge r + A_{i+j-r} \\ B_i + s - j & \ge s - (i + j - r) \end{array} \right. .$$

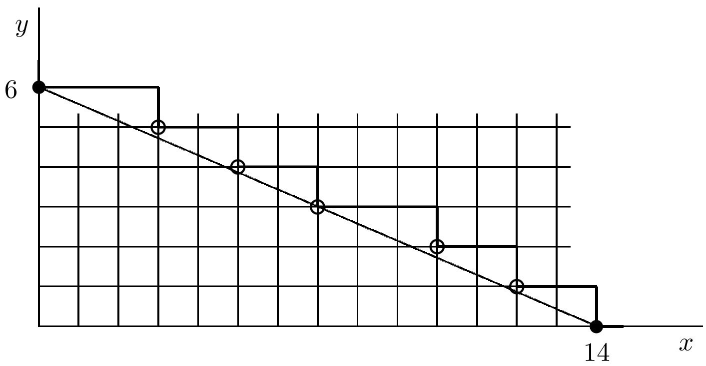
Thus, $x^{i+A_j} y^{B_i+s-j} \in x^r J$.

Finally, the second statement of the proposition follows clearly from the figure. $\square$

According to Proposition 3.3 it will be enough to consider monomial ideals, in which the powers of y decrease by one, in order to determine monomial ideals that are integrally closed.

3.2 Simple integrally closed monomial ideals and factorization

Let $A_r > r$ be positive integers and let $I = \langle y^r, x^{A_1}y^{r-1}, \ldots, x^{A_{r-1}}y, x^{A_r}\rangle = \langle x^{A_i}y^{r-i}\rangle_{0\leq i\leq r},\ A_0 = 0$, be a monomial ideal. We start looking at such ideals in which all the points in $\log I$ lie on or above the line from $(0, r)$ to $(A_r, 0)$. Such an ideal is integrally closed if and only if the set of all integer points on and above this line is equal to $\log I$. We illustrate by an example where $I = \langle y^6, \ldots, x^{14}\rangle$.



Hence, $I = \langle y^6, x^3y^5, x^5y^4, x^7y^3, x^{10}y^2, x^{12}y, x^{14}\rangle$. Note that the staircase corresponding to I consists of two smaller identical staircases $I_1 = \langle y^3, x^3y^2, x^5y, x^7\rangle$ and $I = I_1^{GCD\,(14,6)}$. As we will see further on that is not a coincidence.

In general, the values A_i are easily obtained from $A_i = \lceil i\frac{A_r}{r}\rceil$, where $\lceil z\rceil$ means the least integer that is greater or equal to z. Moreover, if $\mathrm{GCD}(A, B) = d$ then there are $d-1$ points in $\log I$ that will appear *on* the convex line (besides the two end points). Those points divide the staircase corresponding to I into d identical staircases. To classify integrally closed monomial ideals it is sufficient to examine the ideals where A_r and r are relatively prime. It is obvious that there is only one integrally closed monomial ideal, corresponding to a given rational number $\frac{A_r}{r}$, determined in such a way. Those ideals are all special cases of a greater class of monomial ideals that are *simple*, i.e. cannot be written as a product of two proper monomial ideals.

Proposition 3.4. *Let $I = \langle x^{A_i}y^{r-i}\rangle_{0\leq i\leq r} \subset k[x, y]$ where $A_i > i\frac{A_r}{r}$ for $1 \leq i \leq r-1$. Then I is simple as a monomial ideal.*

Proof. Assume that I is a product of two monomial ideals I_1 and I_2, then

$$I_1 = \langle y^i, \ldots, x^a\rangle \text{ and } I_2 = \langle y^{r-i}, \ldots, x^{A_r-a}\rangle \text{ where } 1 \leq i \leq r-1.$$

Since the power product $x^a y^{r-i} \in I$ we must have $a \geq A_i$. The condition on A_i gives in turn

$$A_r - a \leq A_r - A_i < A_r - i\frac{A_r}{r} = A_r(\frac{r-i}{r}). \tag{3.1}$$

On the other hand, as $x^{A_r - a} y^i \in I$ then

$$A_r - a \geq A_{r-i} > (r-i)\frac{A_r}{r} = A_r(\frac{r-i}{r}). \tag{3.2}$$

Since (3.1) and (3.2) contradict each other, the assumption was wrong. □

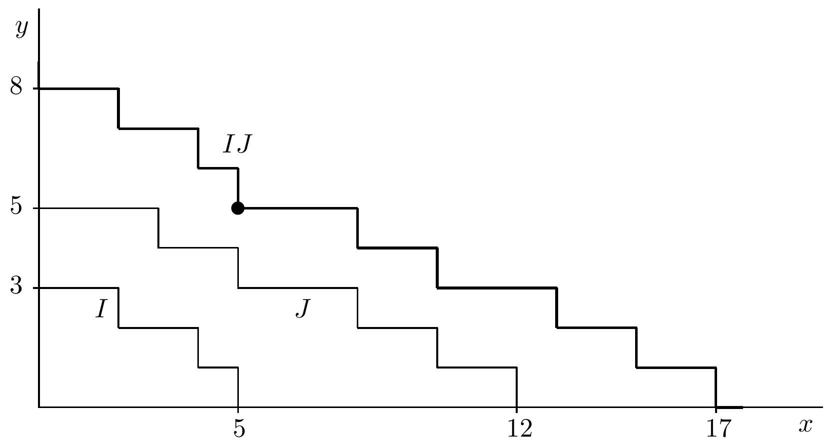
Definition 3.5. Let a and b be a positive integers, such that GCD $(a, b) = 1$. Then there is unique simple integrally closed monomial ideal in $k[x, y]$ possessing x^a and y^b in its minimal generating set. This ideal is called an (a, b)-*block* or a *block ideal*. Moreover, the ideal is the least integrally closed ideal in this class.

In the case when $a > b$ the steps in the staircase corresponding to an (a, b)-block have all height one. If $a < b$ then the steps have breadth one.

Proposition 3.6. *Let I be a (a, b)-block and J a (c, d)-block. Assume further that $\frac{a}{b} \leq \frac{c}{d}$. Then $IJ = y^d I + x^a J$.*

The meaning of the proposition is that the product of two block ideals, will look like the ideal corresponding to the least rational number followed by the other ideal. Note further that IJ is the least integrally closed monomial ideal containing $y^{b+d}, x^a y^d$ and x^{a+c}. We are going to prove the case when $\frac{a}{b}$ and $\frac{c}{d}$ are both greater than one. For two rational numbers both less than one the proof is similar. The case with a rational number less than one and another greater than one is a special case of Proposition 3.3.

Example 3.7. Let $I = \langle y^3, x^2y^2, x^4y, x^5\rangle$ be a $(5, 3)$-block and $J = \langle y^5, x^3y^4, x^5y^3, x^8y^2, x^{10}y, x^{12}\rangle$ a $(12, 5)$-block. Then $IJ = \langle y^8, x^2y^7, x^4y^6, x^5y^5, x^8y^4, x^{10}y^3, x^{13}y^2, x^{15}y, x^{17}\rangle = y^5 I + x^5 J$.



Proof. First we rewrite the proposition in a way that will simplify our reasoning. Let $I = \langle x^{A_i} y^{r-i} \rangle_{0 \leq i \leq r}$ where $A_i = \lceil i \frac{A_r}{r} \rceil$, and $J = \langle x^{C_j} y^{s-j} \rangle_{0 \leq j \leq s}$ where $C_j = \lceil j \frac{C_s}{s} \rceil$. Moreover, let $\frac{A_r}{r} \leq \frac{C_s}{s}$. Then our claim is that $IJ = y^s I + x^{A_r} J$.

Clearly $y^s I + x^{A_r} J \in IJ$.

The ideal IJ is generated by power products on the form

$$x^{A_i + C_j} y^{r+s-i-j} = \begin{cases} x^{A_i+C_j} y^{r-i-j} \cdot y^s & \text{if } i+j \leq r, \\ x^{A_r} \cdot x^{A_i+C_j-A_r} y^{r+s-i-j} & \text{if } i+j \geq r. \end{cases}$$

We will use the following inequalities: $\lceil z_1 \rceil + \lceil z_2 \rceil \geq \lceil z_1 + z_2 \rceil \geq \lceil z_1 \rceil + \lceil z_1 \rceil - 1$, and that $\lceil z_1 + z_2 \rceil = \lceil z_1 \rceil + \lceil z_1 \rceil - 1$ when $z_1 + z_2$ are integers but z_1, z_2 are not.

In the first case we have

$$A_i + C_j \geq A_i + A_j = \lceil i \frac{A_r}{r} \rceil + \lceil j \frac{A_r}{r} \rceil \geq \lceil (i+j) \frac{A_r}{r} \rceil = A_{i+j},$$

so that $x^{A_i+C_j} y^{r-i-j} = x^m x^{A_{i+j}} y^{r-(i+j)} \in I$ for some $m \geq 0$.

In the second case

$$A_i + C_j - A_r \geq A_i - A_r + \lceil (r-i) \frac{C_s}{s} \rceil + \lceil (i+j-r) \frac{C_s}{s} \rceil - 1 \geq$$

$$\geq \lceil i \frac{A_r}{r} \rceil + \lceil (r-i) \frac{A_r}{r} \rceil - 1 - A_r + \lceil (i+j-r) \frac{C_s}{s} \rceil = C_{i+j-r},$$

whence $x^{A_i+C_j-A_r} y^{r+s-i-j} = x^n x^{C_{i+j-r}} y^{s-(i+j-r)} \in J$ for some $n \geq 0$.

In any case $x^{A_i+C_j} y^{r+s-i-j} \in y^s I + x^{A_r} J$. □

We use the last proposition to state the main result of this thesis. By assigning to a (a_k, b_k)-block a rational number $\frac{a_k}{b_k}$ we get a one-to-one correspondence between ascending chains of positive rational numbers and integrally closed monomial ideals in two variables.

Theorem 3.8. *Let $(I_k)_{1 \leq k \leq n} \subset k[x,y]$ be a sequence of (a_k, b_k)-blocks such that $\frac{a_k}{b_k} \leq \frac{a_{k+1}}{b_{k+1}}$. Then the product is an integrally closed ideal*

$$\prod_{k=1}^{n} I_k = \sum_{k=1}^{n} \left(x^{\sum_{k'=1}^{k-1} a_{k'}} y^{\sum_{k'=k+1}^{n} b_{k'}} \right) I_k. \tag{3.3}$$

Conversely, any integrally closed monomial ideal can be written uniquely as a product of block ideals.

Proof. We use induction on n. The statement is valid for $n = 2$ according to Proposition 3.6. Assume that the statement is true for some $n \geq 2$. Then

$$\prod_{k=1}^{n+1} I_k = (\prod_{k=1}^{n} I_k) \cdot I_{n+1} = \left(\sum_{k=1}^{n} x^{\sum_{k'=1}^{k-1} a_{k'}} y^{\sum_{k'=k+1}^{n} b_{k'}} I_k \right) \cdot I_{n+1} =$$

$$\sum_{k=1}^{n} x^{\sum_{k'=1}^{k-1} a_{k'}} y^{\sum_{k'=k+1}^{n} b_{k'}} \left(y^{b_{n+1}} I_k + x^{a_k} I_{n+1}\right) =$$

$$\sum_{k=1}^{n} x^{\sum_{k'=1}^{k-1} a_{k'}} y^{\sum_{k'=k+1}^{n+1} b_{k'}} I_k + \sum_{k=1}^{n-1} x^{\sum_{k'=1}^{k} a_{k'}} y^{\sum_{k'=k+1}^{n} b_{k'}} I_{n+1} + x^{\sum_{k'=1}^{n} a_{k'}} I_{n+1} =$$

$$\sum_{k=1}^{n+1} (x^{\sum_{k'=1}^{k-1} a_{k'}} y^{\sum_{k'=k+1}^{n} b_{k'}}) I_k + \sum_{k=1}^{n-1} x^{\sum_{k'=1}^{k} a_{k'}} y^{\sum_{k'=k+1}^{n} b_{k'}} I_{n+1}.$$

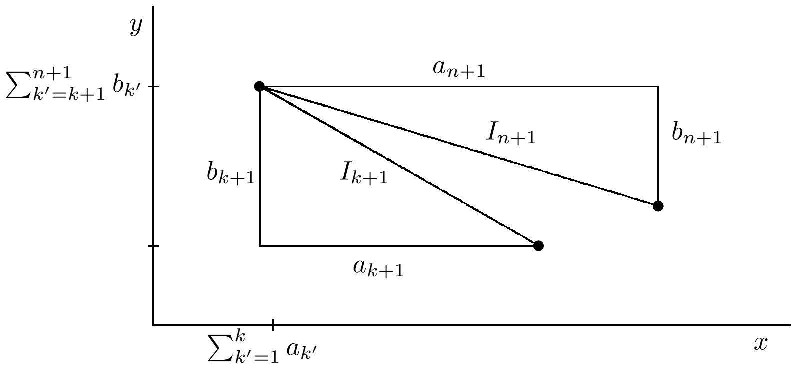
What is left to prove is that the second part in the last row is contained in the first part, which is (3.3) for $n+1$. For each $1 \le k \le n-1$ we can rewrite a term in the second part as follows:

$$\begin{aligned} &x^{\sum_{k'=1}^{k} a_{k'}} y^{\sum_{k'=k+1}^{n} b_{k'}} \langle y^{b_{n+1}}, \ldots, x^{a_{n+1}} \rangle = \\ &\langle x^{\sum_{k'=1}^{k} a_{k'}} y^{\sum_{k'=k+1}^{n+1} b_{k'}}, \ldots, x^{(\sum_{k'=1}^{k} a_{k'})+a_{n+1}} y^{\sum_{k'=k+1}^{n} b_{k'}} \rangle, \end{aligned} \tag{3.4}$$

and compare it with the term corresponding to $k+1$ in the right hand side in (3.3)

$$\begin{aligned} &x^{\sum_{k'=1}^{k} a_{k'}} y^{\sum_{k'=k+2}^{n+1} b_{k'}} \langle y^{b_{k+1}}, \ldots, x^{a_{k+1}} \rangle = \\ &\langle x^{\sum_{k'=1}^{k} a_{k'}} y^{\sum_{k'=k+1}^{n+1} b_{k'}}, \ldots, x^{\sum_{k'=1}^{k+1} a_{k'}} y^{\sum_{k'=k+2}^{n+1} b_{k'}} \rangle. \end{aligned} \tag{3.5}$$

The graphical comparison is depicted below.

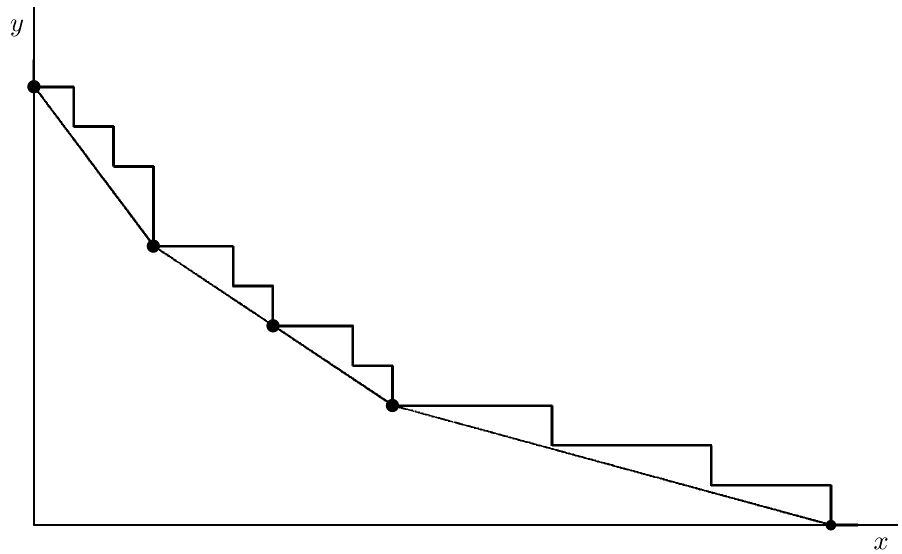


We see that (3.4) is fully contained in (3.5) and possibly the following terms. This proves the first part of the theorem.

Let $I = \langle y^b, \ldots, x^a \rangle$ be an integrally closed monomial ideal. Let $(a_1, b-b_1) \in \log I$ be the point such that there are no other points belonging to $\log I$ below or on the line between $(0,b)$ and $(a_1, b-b_1)$. Then for $k \ge 1$ let $(a_1 + \cdots + a_{k+1}, b - b_1 - \cdots - b_{k+1}) \in \log I$ be the point that satisfies the same condition but for the line between $(a_1 + \cdots + a_k, b - b_1 - \cdots - b_k)$ and $(a_1 + \cdots + a_{k+1}, b - b_1 - \cdots - b_{k+1})$. Determined in such a way and since I is integrally closed, we must have GCD $(a_k, b_k) = 1$. The corresponding (a_k, b_k)-blocks are unique and their product, as described in the first part of the theorem, must be equal to I. □

Example 3.9. Consider the integrally closed ideal $I = \langle y^{11}, xy^{10}, x^2y^9, x^3y^7, x^5y^6, x^6y^5, x^8y^4, x^9y^3, x^{13}y^2, x^{17}y, x^{20}\rangle$.

The points constructed in the way described in the second part of the poof of our theorem are: $(0, 11)$, $(3, 7)$, $(6, 5)$, $(9, 3)$ and $(20, 0)$. Thus $I = I_1I_2I_3I_4$ where $I_1 = \langle y^4, xy^3, x^2y^2, x^3\rangle$, $I_2 = \langle y^2, x^2y, x^3\rangle = I_3$ and $I_4 = \langle y^3, x^4y^2, x^8y, x^{11}\rangle$.



4 Powers of ideals

In the last chapter we could see that the lth power of an integrally closed monomial ideal looks like the staircase, corresponding to the ideal, repeated l times. In a way this result can be extended to certain kinds of ideals in an integral domain. There are two different cases.

4.1 Dividing generators I

Let R be an integral domain and $F(R)$ its field of fractions. Considering $\alpha, \beta \in F(R)$ we say that α divides β, $\alpha \mid \beta$, if there is $p \in R$ such that $\alpha \cdot p = \beta$.

Proposition 4.1. *Let R be an integral domain and $I = \langle r_0, \ldots, r_q\rangle$ an ideal in R, where $r_0 \in R$ and $r_i = r_{i-1}\alpha_i = r_0(\alpha_1 \cdots \alpha_i)$ with $\alpha_i \in F(R)$ and $\alpha_{i-1}|\alpha_i$*

for $1 \le i \le q$. Then for any nonnegative integer l we have

$$\begin{aligned} I^l = \langle\, & r_0^l \quad , r_0^{l-1} r_1, \dots, r_0 r_1^{l-1}, \\ & r_1^l \quad , r_1^{l-1} r_2, \dots, r_1 r_2^{l-1}, \\ & \qquad \vdots \\ & r_{q-1}^l, r_{q-1}^{l-1} r_q, \dots, r_{q-1} r_q^{l-1}, r_q^l \rangle = \\ = \langle\, & r_i^{l-t} r_{i+1}^t \ ;\ 0 \le i \le q-1 \text{ and } 0 \le t \le l \rangle. \end{aligned} \tag{4.1}$$

Proof. We get immediately that $\langle\, r_i^{l-t} r_{i+1}^t \rangle \subseteq I^l$.

To show the other inclusion we need the following remark.

Remark 4.2. Pick some r_i and $r_{i'}$ with $i < i'$, then the product $r_i r_{i'} = (r_{i+1}/\alpha_{i+1}) \cdot r_{i'-1}\alpha_{i'} = r_{i+1} r_{i'-1}(\alpha_{i'}/\alpha_{i+1})$. Since $\alpha_{i+1} \mid \alpha_{i'}$ we have $r_{i+1} r_{i'-1} \mid r_i r_{i'}$. In the language of ideals this means $\langle r_i r_{i'} \rangle \subseteq \langle r_{i+1} r_{i'-1} \rangle$.

We know that I^l is generated by the elements $r_0^{l_0} \cdots r_q^{l_q}$, where $\sum_{i=0}^q l_i = l$. Then, if $l_0 \le l_q$, the ideal I^l lies in the ideal in which one of the generators is replaced by the element $r_1^{l_1+l_0} \cdots r_{q-1}^{l_{q-1}+l_0} r_q^{l_q-l_0}$ (or by $r_0^{l_0-l_q} r_1^{l_1+l_q} \cdots r_{q-1}^{l_{q-1}+l_q}$ if $l_0 \ge l_q$). Repeating the procedure of replacements for this generator we can eventually "replace" it by r_i^l or $r_i^{l-t} r_{i+1}^t$ for some i and t. Doing the same for all the other generators, we see that $I^l \subseteq \langle\, r_i^{l-t} r_{i+1}^t \rangle$. This finishes the proof. □

Remark 4.3. It is easily seen that I^l itself satisfies the conditions on I in the proposition.

We apply the conditions in the proposition on $R = k[x, y]$ and I an $\langle x, y \rangle$-primary monomial ideal. Let $I = \langle m_0, \dots, m_q \rangle$ where $m_0 = y^b$ is the power product with highest y-exponent, while $\alpha_i = \frac{x^{a_i}}{y^{b_i}}$ where $a_i \le a_{i+1}$, $b_i \ge b_{i+1}$ and $\sum_{i=1}^q b_i = b$. Then we can rewrite $m_0 = y^{b_1+\cdots+b_q}$, $m_i = x^{a_1+\cdots+a_i} y^{b_{i+1}+\cdots+b_q}$ for $1 \le i \le q-1$ and $m_q = x^{a_1+\cdots+a_q}$. These ideals can be factorized in a very simple way.

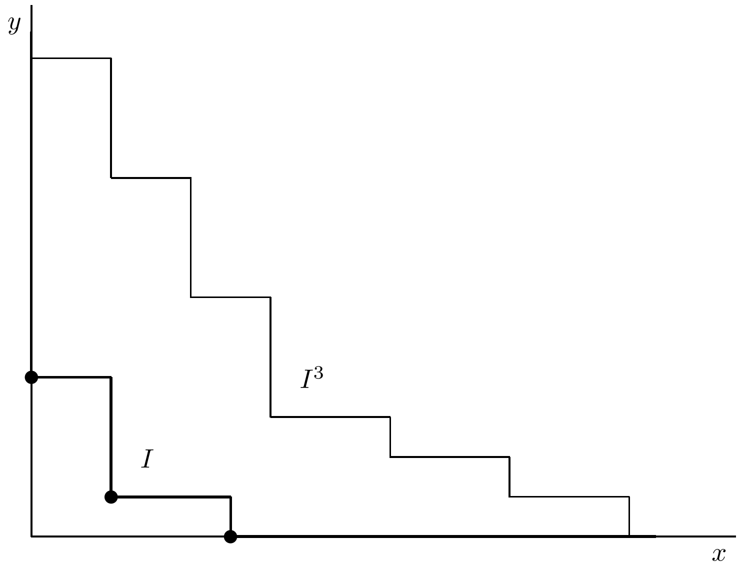
Proposition 4.4. *Let $R = k[x, y]$ and $I = \langle m_0, \dots, m_q \rangle$ be a monomial ideal where m_i's satisfy the conditions on generators in Proposition 4.1. Further, we may assume that $m_0 = y^b$ and $\alpha_i = \frac{x^{a_i}}{y^{b_i}}$ where $a_i \le a_{i+1}$, $b_i \ge b_{i+1}$ and $\sum_{i=1}^q b_i = b$. Then*

$$I = \prod_{i=1}^q \langle x^{a_i}, y^{b_i} \rangle. \tag{4.2}$$

Proof. Obviously m_0, m_q and $m_i = x^{a_1+\ldots+a_i} y^{b_{i+1}+\ldots+b_q} \in \prod_{i=1}^q \langle x^{a_i}, y^{b_i} \rangle$.

Conversely, the right hand side of (4.2) is generated by elements of the following type, $x^{a_{i_1}+\ldots+a_{i_r}} y^{b_{j_1}+\ldots+b_{j_s}}$ where $r + s = q$, which are divisible by the element $x^{a_1+\ldots+a_r} y^{b_{r+1}+\ldots+b_q}$ belonging to I, and we are done. □

We continue with how powers of I look like. Considering I as a staircase and going down-wards, its steps are increasing in breadth and decreasing in height. The lth power makes a staircase where the first step in I is repeated l times followed by the second repeated l times and so on.



4.2 Dividing generators II.

We state a proposition similar to Proposition 4.1 but for another kind of ideals.

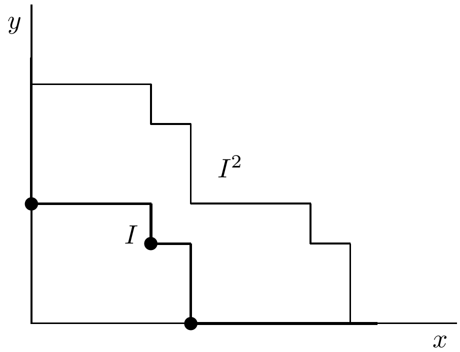
Proposition 4.5. *Let R be an integral domain and $I = \langle s_0, \dots, s_q \rangle$ be an ideal in R, where $s_0 \in R$ and $s_i = s_{i-1}\beta_i = s_0(\beta_1 \cdots \beta_i)$ with $\beta_i \in F(R)$ and $\beta_i \mid \beta_{i-1}$ for $1 \le i \le q$. Then for any nonnegative integer l we have*

$$\begin{aligned} I^l = \langle\, & s_0^l, s_0^{l-1}s_1, \dots, s_0^{l-1}s_{q-1}, \\ & s_0^{l-1}s_q, s_0^{l-2}s_1s_q, \dots, s_0^{l-2}s_{q-1}s_q, \\ & \vdots \\ & s_0s_q^{l-1}, s_1s_q^{l-1}, \dots, s_{q-1}s_q^{l-1}, s_q^l \rangle = \\ = \langle\, & s_0^{l-t}s_is_q^{t-1}\ ;\ 0 \le i \le q \text{ and } 1 \le t \le l \rangle. \end{aligned} \tag{4.3}$$

Proof. Clearly, $\langle\, s_0^{l-t}s_is_q^{t-1} \rangle \subseteq I^l$.

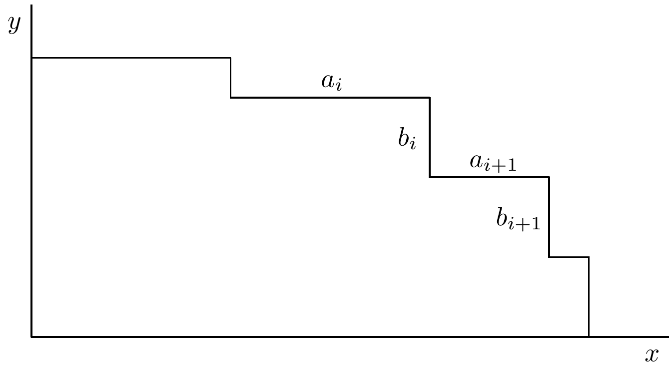
For proving the other inclusion, we notice at first that for any s_i and $s_{i'}$ with $1 \le i \le i' \le q-1$ we have that the product $s_is_{i'} = (s_{i-1}\beta_i) \cdot (s_{i'+1}/\beta_{i'+1}) = s_{i-1}s_{i'+1}(\beta_i/\beta_{i'+1})$. This means $\langle s_is_{i'} \rangle \subseteq \langle s_{i-1}s_{i'+1} \rangle$, since $\beta_{i'+1} \mid \beta_{i'} \mid \beta_i$. Further, for any generator in I^l we can therefore apply the following procedure: as long as $\sum_{i=1}^{q-1} l_i \ge 2$ in a generator $s_0^{l_0} \cdots s_j^{l_j} \cdots s_{j'}^{l_{j'}} \cdots s_q^{l_q}$, we can replace it by the element $s_0^{l_0} \cdots s_{j-1} s_j^{l_j - 1} \cdots s_{j'}^{l_{j'}-1} s_{j'+1} \cdots s_q^{l_q}$. Repeating the procedure until $\sum_{i=1}^{q-1} l_i \leq 1$, we see that $I^l \subseteq \langle\, s_0^{l-t} s_i s_q^{t-1} \rangle$. □

When $R = k[x, y]$ and I monomial, we see that, in accordance with the case described in Subsection 4.1, we may express the ideal as $I = \langle m_0, \dots, m_q \rangle$ where $m_0 = y^b$ and $\beta_i = \frac{x^{a_i}}{y^{b_i}}$ with $a_i \geq a_{i+1}$ and $b_i \leq b_{i+1}$. Its steps are decreasing in breadth and increasing in height. The lth power is the whole staircase I repeated l times.



From the figure we see directly that I^l does *not* generally fulfill the conditions on I. Furthermore, a monomial ideal of such type cannot be written as a product of two monomial ideals. We state that fact in a proposition.

Proposition 4.6. *Let* $I = \langle y^{b_1 + \cdots + b_q}, \dots, x^{a_1 + \cdots + a_i} y^{b_{i+1} + \cdots + b_q}, \dots, x^{a_1 + \cdots + a_q} \rangle \subset k[x, y]$ *where* $a_i \geq a_{i+1}$ *and* $b_i \leq b_{i+1}$ *with strict inequality occures at least once. Then* I *is simple as monomial ideal.*



Proof. Assume that strict inequality occurs for some index among exponents of y.

If I is a product of two monomial ideals J_1 and J_2, then

$$J_1 = \langle y^{b'}, \dots, x^{a'} \rangle, \text{ where } b' + b'' = b_1 + \cdots + b_q \text{ and}$$
$$J_2 = \langle y^{b''}, \dots, x^{a''} \rangle, \text{ where } a' + a'' = a_1 + \cdots + a_q.$$

For each $b' < b_1 + \cdots + b_q$ that we may choose, there exists some $1 \le i \le q-1$ such that $b_{i+1} + \cdots + b_q \le b' < b_i + \cdots + b_q$ (we do not let $0 \le b' < b_q$ since it would automatically give $a'' \ge a_1 + \cdots + a_q$, in order to have $x^{a''}y^{b'} \in I$, and $a' < 0$ therefore). This condition implies on one hand that $b_1 + \cdots + b_{i-1} < b'' \le b_1 + \cdots + b_i$, on the other hand that $a'' \ge a_1 + \cdots + a_i$ in order to have $x^{a''}y^{b'} \in I$ which implies in turn that $a' \le a_{i+1} + \cdots + a_q$.

Let us now consider the power product $x^{a'}y^{b''} \in J_1J_2$. We have $b'' \le b_1 + \cdots + b_i < b_{q-i+1} + \cdots + b_q$, since we assumed strict inequality somewhere, and hence $a' \ge a_1 + \cdots + a_{q-i+1}$ so that $x^{a'}y^{b''} \in I$. Together with the previous condition on a' that we got, we have

$$a_1 + \cdots + a_{q-i+1} \le a' \le a_{i+1} + \cdots + a_q,$$

which is a contradiction because the number of terms on the left is greater than on the right and $a_i \ge a_{i+1}$. Thus, the assumption that I could be a product of two monomial ideals was false. □

References

[1] M. Herrmann, S. Ikeda, U. Orbanz, *Equimultiplicity and Blowing up*, Springer-Verlag, 1988.

[2] M. Hochster, C. Huneke, J.D. Sally, *Commutative Algebra: proceedings from a microprogram*, Springer-Verlag, 1989.

[3] R. Villarreal, *Monomial Algebras*, Marcel Dekker, 2001.

[4] O. Zariski, P. Samuel, *Commutative Algebra. Vol. 2*, Princeton, 1960.