A unified approach for the H_∞-stability analysis of classical and fractional neutral systems with commensurate delays

Le Ha Vy Nguyen

A UNIFIED APPROACH FOR THE H_∞-STABILITY ANALYSIS OF CLASSICAL AND FRACTIONAL NEUTRAL SYSTEMS WITH COMMENSURATE DELAYS

LE HA VY NGUYEN*

Abstract. We examine the stability of linear integer-order and fractional-order systems with commensurate delays of neutral type in the sense of H_∞-stability. The systems may have chains of poles approaching the imaginary axis. While several classes of these systems have been previously studied on a case-by-case basis, a unified method is proposed in this paper which allows to deal with all these classes at the same time. Approximation of poles of large modulus is systematically calculated based on a convex hull derived from the coefficients of the system. This convex hull also serves to establish sufficient conditions for instability and necessary and sufficient conditions for stability.

Key words. Neutral systems, delay effects, H_∞-stability, fractional systems

AMS subject classifications. 93D05, 93D25, 93C05

1. Introduction. Due to the ubiquitous presence of time delays, integer-order systems with time delays have been intensively studied (see, for example, [3, 25, 21, 14] and the references therein). Recently, fractional-order systems with time delays have become an active research subject [13, 11, 5, 10, 20, 26] because of the increasing use of fractional models. We can find these models, for example, in electronics such as capacitors [28, 8] and in biology such as human, animal, and plant tissues [9]. For some background on fractional calculus, see [19, 23, 12].

In this paper, we consider linear classical and fractional systems with commensurate delays. For classical delay systems, since the seminal work of Bellman and Cooke [3], it is well known that these systems can be classified in three categories regarding the location of poles of large modulus: advanced, retarded and neutral types. A similar classification is applied for fractional systems with commensurate delays and commensurate fractional orders [11, 6].

While advanced systems having infinitely many unstable poles of large modulus are unstable, the stability of retarded systems is decided by the location of poles of small modulus [3, 5]. They can be calculated by several numerical methods, for example, QPmR [27], YALTA [2].

Neutral systems having chains of poles asymptotic to vertical lines strictly in the right or left half-planes fall in the same stability scenario as advanced or retarded systems. It remains the case where the asymptotic line is the imaginary axis. For these systems to be stable, it is not only required that poles of small and large modulus are in the open left half-plane, but other conditions are also needed (see, for example, [1] for BIBO-stability and [22, 6] for H_∞-stability). The situation is similar in the time domain: sufficient conditions for asymptotic stability include the classical condition 'all eigenvalues have negative real part' and we can find in [24] a deep stability analysis of neutral systems containing fixed and distributed delays. However, the links between stability conditions for this class of neutral systems in the frequency and time domain are not fully understood.

For these special systems, while poles of small modulus can easily be numerically determined as in the retarded case, poles of large modulus have been approximated

---

*Inria, Université Paris-Saclay, L2S-CentraleSupelec, 3 Rue Joliot-Curie, 91192 Gif-sur-Yvette, France (nguyenlehavy@gmail.com).

for some classes of systems [4, 18, 16, 17]. The approximations were then examined and necessary and sufficient conditions for H_∞-stability were given when appropriate.

Although the conclusions on the location of chains of poles about the imaginary axis were different for different classes of systems, the general method used for approximating poles in neutral chains remained the same. However, establishing the results became more complicated for classes of systems which require higher order approximation.

To overcome this difficulty, we provide in this paper new results which generalize those of the previous works and which can be easily implemented in computation software. They cover both classical and fractional systems in almost every configuration.

The paper is organized as follows. Section 2 presents the (fractional) neutral delay systems of interest. The main results concerning the location of poles and stability conditions are presented in section 3 and section 4 respectively. In section 5, these results are compared with some of those presented in [16, 17]. They are also applied to a class of systems that has not been considered in the previous works. The paper is then concluded by section 6.

We denote $\mathbb{N}_N = \{n \in \mathbb{N} \,|\, 1 \le n \le N\}$, $\mathbb{Z}_+ = \{m \in \mathbb{Z} \,|\, m \ge 0\}$, $\mathbb{C}_+ = \{s \in \mathbb{C} \,|\, \Re(s) > 0\}$, $\ln(\cdot)$ the real logarithm function, $\mathrm{Arg}(s)$ the argument of $s \in \mathbb{C}$ satisfying $-\pi < \mathrm{Arg}(s) \le \pi$, $[x]$ the integer part of $x \in \mathbb{R}$, and $n(A)$ the number of elements of a set A.

2. A class of (fractional) neutral time-delay systems. We consider (fractional) neutral time-delay systems described by transfer functions of the form

$$(1) \qquad G(s) = \frac{t(s)}{p(s) + \sum\limits_{k=1}^{N} q_k(s) e^{-ks\tau}},$$

where $\tau > 0$ is the delay, t, p, and q_k for all $k \in \mathbb{N}_N$ are real polynomials in s^μ with $0 < \mu \le 1$, $-\pi < \arg(s) \le \pi$ in the case where $0 < \mu < 1$ in order to have a single value of s^μ, $\deg p \ge \deg t$, $\deg p \ge \deg q_k$ for all $k \in \mathbb{N}_N$, and $\deg p = \deg q_k$ at least for one $k \in \mathbb{N}_N$ in order to deal with proper neutral systems. Here, the degree of a (quasi-)polynomial refers to the degree in s^μ.

Note that with $\mu \in (0, 1]$, the systems defined by (1) encompass those studied in [4, 18, 16, 17].

Here we recall some basic characteristics of these systems presented in [4, 18, 16, 17].

Since $\deg p \ge \deg q_k$ for all $k \in \mathbb{N}_N$, then for each k we obtain

$$(2) \qquad \frac{q_k(s)}{p(s)} = \alpha_{0,k} + \sum_{l=1}^{M'} \frac{\alpha_{l,k}}{s^{l\mu}} + \mathcal{O}(s^{-(M'+1)\mu})$$

as $|s| \to \infty$, where $M' \in \mathbb{Z}_+$ and can be arbitrarily large.

Remark 2.1. In this paper, for the purpose of developing more general results, we change some notations compared to [4, 18, 16, 17]. In the development of $q_k(s)/p(s)$ for $k \in \mathbb{N}_N$ as $|s| \to \infty$, the coefficients corresponding to the terms $s^{-l\mu}$ for $l \in \mathbb{Z}_+$ are now denoted by $\alpha_{l,k}$. Hence $\alpha_{0,k}$, $\alpha_{1,k}$, $\alpha_{2,k}$, $\alpha_{3,k}$, and $\alpha_{4,k}$ replace respectively α_k, β_k, γ_k, δ_k, and ϵ_k in the previous papers.

The formal polynomial associated with the denominator of (1) is defined as

$$\widetilde{c}_d(z) := 1 + \sum_{k=1}^{N} \alpha_{0,k} z^k,$$

(3)

where $z = e^{-s\tau}$.

To each root r of multiplicity $m \geq 1$ of (3) is associated m neutral chain(s) of poles of G. These poles, denoted by s_n with $n \in \mathbb{Z}$, are approximated by

(4)
$$s_n \tau = \lambda_n + o(1),$$

where

(5)
$$\lambda_n = -\ln|r| - \jmath \operatorname{Arg}(r) + \jmath 2\pi n,$$

as $n \to \infty$.

It is derived from (4) that the poles of the neutral chain asymptotically approach the vertical line

$$\Re(s) = -\frac{\ln|r|}{\tau}.$$

When the vertical line is different from the imaginary axis, i.e., when $|r| \neq 1$, poles of large modulus clustering this line are on the same side w.r.t. the imaginary axis and thus can be classified as stable (i.e., $|r| > 1$) or unstable (i.e., $|r| < 1$) poles regarding H_∞-stability.

Recall that a system is H_∞-stable if and only if its transfer function is analytic and bounded in $\mathbb{C}_+$.

We recall the following lemma which will be used when considering multiple roots of $\widetilde{c}_d(z)$.

LEMMA 2.2 ([17, Lemma A.1]).   *Let r be a root of multiplicity $m > 1$ of $f(z) = 1 + \sum_{k=1}^{N} \alpha_k z^k$, where $\alpha_k \in \mathbb{C}$. Then $\sum_{k=1}^{N} k^l \alpha_k r^k = 0$ for $l = 1, \ldots, m-1$ and $\sum_{k=1}^{N} k^m \alpha_k r^k \neq 0$.*

3. Location of neutral poles. As we have seen in the previous section, to each root r of multiplicity $m \geq 1$ of the formal polynomial $\widetilde{c}_d(z)$ correspond m chain(s) of poles of neutral type. The approximation of these poles given in (4) only indicates the vertical line to which the pole chain approaches. To determine the position of the chain around the asymptotic axis, similarly to [4, 18, 16, 17], we examine in this section a more precise approximation of neutral poles of the form

$$s_n \tau = \lambda_n + \nu_{n,1} + o(n^{-y_1})$$

with

$$\nu_{n,1} = \frac{\nu_1}{n^{y_1}}, \ \nu_1 \neq 0, y_1 > 0, n \in \mathbb{Z}, n \to \infty, -\pi < \arg(n) \leq \pi.$$

In other words, we determine the next *non-zero* approximation term when it is appropriate. Such an approximation term does not exist if the neutral poles are precisely $s_n = \lambda_n/\tau$.

Except that special case, $\nu_{n,1}$ exists and the sign of $\Re(\nu_1/n^{y_1})$ then shows on which side of the asymptotic axis the poles are. Note that the sign may change for

positive and negative n. Hence, the upper and lower parts of a pole chain may lie on different sides of the asymptotic axis. For a detailed explanation, see [17, Remark 3].

Here, remark that we do not fix a value of y_1 beforehand but look for y_1 such that $\nu_1 \neq 0$. This ensures that the approximation gives some new information about the location of poles. The only case where the information is not useful is when $\Re(\nu_1/n^{y_1}) = 0$ and we may need to approximate further to know the location of poles about the asymptotic axis.

3.1. Approximation of neutral chains of poles. Before presenting the main results about the location of poles around the asymptotic axis, we define some notions which will be of use.

DEFINITION 3.1. *For a root r of $\widetilde{c}_d(z)$,*

$$(6) \qquad AB(r) := \{(a,b) \in \mathbb{Z}_+^2 \mid a+b \neq 0, \sum_{k=1}^{N} \alpha_{a,k} k^b r^k \neq 0\},$$

where $\alpha_{a,k}$, $a \in \mathbb{Z}_+$ and $k = 1, \ldots, N$, are given in (2).

Remark 3.2. The point $(0, m)$, where m is the multiplicity of the root r, belongs to $AB(r)$ since $\sum_{k=1}^{N} \alpha_{0,k} k^m r^k \neq 0$ (see Lemma 2.2). From the same lemma, we see that the points $(0, l)$ with $0 \leq l < m$ are not in $AB(r)$.

DEFINITION 3.3.
- *A* lower left boundary segment *of $AB(r)$ is a subset $\mathcal{S} \subset AB(r)$ such that $n(\mathcal{S}) \geq 2$ and there exists $\mathfrak{p} > 0$ such that $a + b\mathfrak{p} = a' + b'\mathfrak{p} \ \forall(a,b), (a',b') \in \mathcal{S}$ and $a + b\mathfrak{p} < a'' + b''\mathfrak{p} \ \forall(a'',b'') \in AB(r)\backslash\mathcal{S}$.*
- *$\mathfrak{p}$ defined as above for $\mathcal{S}$ is obviously unique and we call it the* inverse slope *of the segment. We will use the notation $\mathfrak{p}(\mathcal{S})$ to denote the inverse slope of a lower left boundary segment $\mathcal{S}$.*
- *$\mathfrak{S}(AB(r))$ denotes the set of all lower left boundary segments of $AB(r)$.*

It is easy to prove that different lower left boundary segments have different inverse slopes, i.e., for $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{S}(AB(r))$, $\mathcal{S}_1 \neq \mathcal{S}_2$, their inverse slopes satisfy $\mathfrak{p}(\mathcal{S}_1) \neq \mathfrak{p}(\mathcal{S}_2)$.

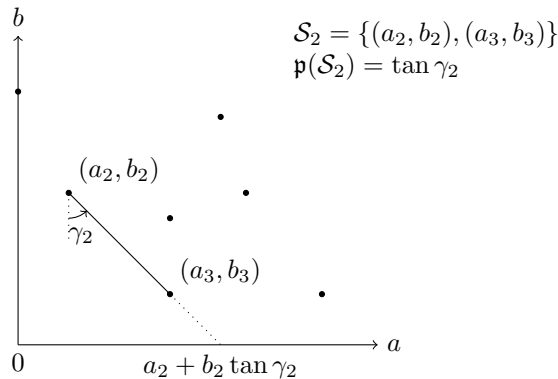A lower left boundary segment $\mathcal{S}_2$ is illustrated in Figure 1.

Fig. 1: A lower left boundary segment of a set of points in the plane

The approximation of neutral chains of poles is the objective of the next theorem. Note that among m neutral chains associated to the root r of multiplicity m, some may have poles satisfying $s_n = \lambda_n/\tau$ for $n \in \mathbb{Z}$, $n \to \infty$, where λ_n is given by (5). The approximation detailed below is thus not applied to these chains of poles.

THEOREM 3.4. *Let $G(s)$ be a neutral delay system defined by* (1) *and r a root of multiplicity m of the formal polynomial $\widetilde{c}_d(z)$ defined by* (3). *With $\alpha_{a,k}$ defined as in* (2), *let us define*

$$C(a,b,\nu) := \frac{\tau^{a\mu}}{(\jmath 2\pi)^{a\mu}} \frac{(-1)^b \nu^b}{b!} \sum_{k=1}^{N} \alpha_{a,k} k^b r^k,$$

$$B(\mathcal{S}) := \left\{ (\nu, y) \mid \nu \text{ is a non-zero root of } \sum_{(a,b) \in \mathcal{S}} C(a,b,\nu) = 0, \ y = \mathfrak{p}(\mathcal{S})\mu \right\}.$$

Let us denote m_a the number of chains of poles relative to r with poles of large modulus on the asymptotic axis given by $s_n = \lambda_n/\tau$ where $n \in \mathbb{Z}$, $n \to \infty$ and λ_n is given by (5). *Then poles of the other neutral chains corresponding to r are approximated by*

(7)
$$s_n = \frac{1}{\tau}\left(\lambda_n + \frac{\nu_1}{n^{y_1}}\right) + o(n^{-y_1})$$

where for each chain of poles (ν_1, y_1) takes one of the $m - m_a$ values (counting multiplicities) given by

$$(\nu_1, y_1) \in \bigcup_{\mathcal{S} \in \mathfrak{S}(AB(r))} B(\mathcal{S})$$

with $AB(r)$ defined in (6).

For the sake of clarity, the proof of the theorem will be preceded by lemmas. The first step is to develop the denominator of $G(s)$ around poles of neutral chains, which is the objective of Lemma 3.5. Next, due to the fact that the denominator is zero at these poles, by examining the highest order in n of its development, which will be done in Lemma 3.6, we derive the approximation term $\nu_{n,1}$ of s_n.

LEMMA 3.5. *Let $d(s)$ be the denominator of $G(s)$, i.e.,*

$$d(s) := p(s) + \sum_{k=1}^{N} q_k(s) e^{-ks\tau}.$$

Assume poles in neutral chains related to r have the form

(8)
$$s_n \tau = \lambda_n + \nu_{n,1} + \nu_{n,2} + \ldots + \nu_{n,M} + o(n^{-M'\mu}),$$

where $n \in \mathbb{Z}$, $\nu_{n,i} = \nu_i n^{-y_i}$, $i = 1, \ldots, M$ with $\nu_i \neq 0$ and $0 < y_1 < \ldots < y_M \leq M'\mu$, $M' \in \mathbb{Z}_+$. Then the development of $\frac{d(s_n)}{p(s_n)}$ up to order $-M'\mu$ as $n \to \infty$ is given as follows

(9)
$$\frac{d(s_n)}{p(s_n)} = g_1(n) + g_2(n) + g_3(n) + o(n^{-M'\mu}) = 0,$$

where

$$g_1(n) := \sum_{l=1}^{M'} \frac{\tau^{l\mu}}{(j2\pi n)^{l\mu}} \left(1 + O(n^{-1})\right) \sum_{k=1}^{N} \alpha_{l,k} r^k,$$

$$g_2(n) := \sum_{(l_1,\ldots,l_M)\in\mathcal{L}(M'\mu)} \frac{(-1)^{\sum_{i=1}^{M} l_i} \left(\prod_{i=1}^{M} \nu_i^{l_i}\right)}{\left(\prod_{i=1}^{M} l_i!\right) n^{\sum_{i=1}^{M} l_i y_i}} \sum_{k=1}^{N} \alpha_{0,k} r^k k^{\sum_{i=1}^{M} l_i},$$

$$g_3(n) := \sum_{(l,l_1,\ldots,l_M)\in\mathcal{L}_0(M'\mu)} \frac{\tau^{l\mu}}{(j2\pi n)^{l\mu}} \left(1 + O(n^{-1})\right) \frac{(-1)^{\sum_{i=1}^{M} l_i} \left(\prod_{i=1}^{M} \nu_i^{l_i}\right)}{\left(\prod_{i=1}^{M} l_i!\right) n^{\sum_{i=1}^{M} l_i y_i}}$$

$$\times \sum_{k=1}^{N} \alpha_{l,k} r^k k^{\sum_{i=1}^{M} l_i}$$

with

$$\mathcal{L}(x) := \left\{ (l_1,\ldots,l_M) \mid l_i \in \mathbb{Z}_+, \sum_{i=1}^{M} l_i \geq 1, \text{ and } \sum_{i=1}^{M} l_i y_i \leq x \right\},$$

$$\mathcal{L}_0(x) := \left\{ (l,l_1,\ldots,l_M) \mid l \in \mathbb{Z}_+\backslash\{0\}, l_i \in \mathbb{Z}_+, \sum_{i=1}^{M} l_i \geq 1, l\mu + \sum_{i=1}^{M} l_i y_i \leq x \right\}.$$

The proof of the lemma is given in Appendix A.

Now we characterize the highest order of the development presented in Lemma 3.5.

LEMMA 3.6. *The highest order in n of the development of $\frac{d(s_n)}{p(s_n)}$ as $n \to \infty$ given as in* (9) *has the form* $-(a\mu + by_1)$ *where* $(a,b) \in AB(r)$.

Proof. The highest order in n of the development (9) can be determined among the highest orders in n of $g_1(n)$, $g_2(n)$, and $g_3(n)$.

The highest order in n of $g_1(n)$ has the form $-a\mu$ where $(a,0) \in AB(r)$.

The orders in n of $g_2(n)$ have the form $-\sum_{i=1}^{M} l_i y_i$ with $(l_1,\ldots,l_M) \in \mathcal{L}(M'\mu)$. Note that $\sum_{i=1}^{M} l_i y_i \geq \sum_{i=1}^{M} l_i y_1$. Hence by definition $(\sum_{i=1}^{M} l_i, 0, \ldots, 0) \in \mathcal{L}(M'\mu)$. Therefore, the highest order in n of $g_2(n)$ has the form $-by_1$ with $(0,b) \in AB(r)$.

By similar arguments, we deduce that the highest order in n of $g_3(n)$ has the form $-(a\mu + by_1)$ with $(a,b) \in AB(r)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Below we prove Theorem 3.4 using the results of Lemmas 3.5 and 3.6.

Proof of Theorem 3.4. Let us denote Y_1 the set of all valid values of y_1 (there are several chains of poles associated to a multiple root r and different chains may have different values of y_1). By examining the term of highest order in n of the development (9), we will prove that $Y_1 = \mu M(AB(r))$ where $M(AB(r)) := \{\mathfrak{p}(\mathcal{S}) \mid \mathcal{S} \in \mathfrak{S}(AB(r))\}$.

First, we prove that $Y_1 \subset \mu M(AB(r))$.

From Lemma 3.6, the term of highest order in n of the development which corresponds to $y_1 \in Y_1$ is then $\sum_{(a,b)\in\widetilde{\mathcal{S}}(y_1)} C(a,b,\nu_1)/n^{a\mu+by_1}$ where the subset $\widetilde{\mathcal{S}}(y_1)$ of $AB(r)$ is defined as follows: $a\mu + by_1 = a'\mu + b'y_1 \ \forall (a,b), (a',b') \in \widetilde{\mathcal{S}}(y_1)$ and $a\mu + by_1 < a''\mu + b''y_1 \ \forall (a'',b'') \in AB(r)\backslash\widetilde{\mathcal{S}}(y_1)$. The latter inequality is due to the fact that $-(a\mu + by_1)$ is the highest order in n.

Now we prove that $\widetilde{\mathcal{S}}(y_1)$ is a lower left boundary segment of $AB(r)$. Assume that $n(\widetilde{\mathcal{S}}(y_1)) = 1$ and that $\widetilde{\mathcal{S}}(y_1) = \{(a,b)\}$. The term of highest order in n of the

development is then $C(a,b,\nu_1)/n^{a\mu+by_1}$. Hence, due to (9), $C(a,b,\nu_1) = 0$ and thus $\nu_1 = 0$ which does not satisfy the assumptions about $\nu_{n,1}$ (i.e., $\nu_{n,1} = \nu_1/n^{y_1}$ with $\nu_1 \neq 0$ and $y_1 > 0$). Therefore, $n(\widetilde{\mathcal{S}}(y_1)) \geq 2$ and thus $\widetilde{\mathcal{S}}(y_1)$ is a lower left boundary segment of $AB(r)$.

Hence, $y_1/\mu = \mathfrak{p}(\widetilde{\mathcal{S}}(y_1))$ and thus $y_1/\mu \in M(AB(r))$, that is $y_1 \in \mu M(AB(r))$. Therefore, $Y_1 \subset \mu M(AB(r))$.

Next we prove that $\mu M(AB(r)) \subset Y_1$.

Assume that $y_1 = \mathfrak{p}\mu$ where $\mathfrak{p} \in M(AB(r))$. Denote $\mathcal{S} \in \mathfrak{S}(AB(r))$ the segment associated with $\mathfrak{p}$. The highest order in n of the development is then $-(a\mu + by_1)$ $\forall (a,b) \in \mathcal{S}$. The term of highest order in n is $\sum_{(a,b)\in\mathcal{S}} C(a,b,\nu_1)/n^{a\mu+by_1}$. Since $n(\mathcal{S}) \geq 2$ by definition, this sum contains terms with different powers in ν_1. Noting that this sum is zero, we then obtain non-zero values of ν_1, which satisfies the assumptions about $\nu_{n,1}$.

Hence, $Y_1 = \mu M(AB(r))$. $\square$

3.2. Construction of lower left boundary segments. Now we will discuss how to construct all the lower left boundary segments of the set $AB(r)$.

First, we mention two important points of $AB(r)$ which limits a subset of $AB(r)$ containing the lower left boundary segments. The first point is $(0, m)$ where m is the multiplicity of the root r. This point belongs to $AB(r)$ since $\sum_{k=1}^{N} \alpha_{0,k} k^m r^k \neq 0$ (see Lemma 2.2). The second point, denoted by (a_L, b_L), is the leftmost point among the lowest points of $AB(r)$, i.e.,

$$
\begin{aligned}
b_L &:= \min\{b \mid (a,b) \in AB(r)\}, \\
a_L &:= \min\{a \mid (a, b_L) \in AB(r)\}.
\end{aligned}
\tag{10}
$$

The limiting finite subset is presented in the next lemma.

LEMMA 3.7. *The lower left boundary segments of $AB(r)$ belong to the subset $A_L^m = \{(a,b) \in AB(r) \mid a \leq a_L, b \leq m\}$ (see Figure 2).*

Proof. If $(a,b) \in AB(r)$ and $a > a_L$, then $a + b\mathfrak{p} > a_L + b_L\mathfrak{p}$ for all $\mathfrak{p} > 0$ since $b \geq b_L$ by definition. If $(a,b) \in AB(r)$ and $b > m$, then $a + b\mathfrak{p} > m\mathfrak{p}$ for all $\mathfrak{p} > 0$ since $a \geq 0$ by definition. $\square$
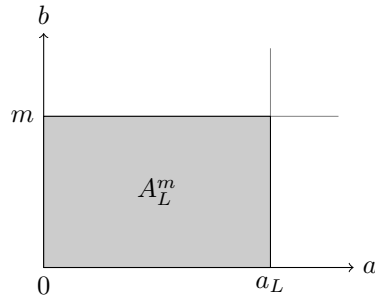

Fig. 2: The subset A_L^m of $AB(r)$ which contains all lower left boundary segments of $AB(r)$

The lower left boundary segments are determined from the limiting subset as follows.

LEMMA 3.8. *The points of a lower left boundary segment of $AB(r)$ belong to an edge of the convex hull of A_L^m and two of them are vertices of the hull.*

Proof. The subset A_L^m has finite points and thus its convex hull is a convex polygon [7]. The vertices of this polygon are points in A_L^m and the line containing each of its edges defines a closed half-plane containing all the points of A_L^m. There is no other line containing two points of A_L^m with such a characteristic.

Therefore, the definition of lower left boundary segments of $AB(r)$ leads immediately to the conclusion. □

There exist numerous algorithms for determining the points of a finite set in $\mathbb{R}^2$ which are on the boundary of its convex hull [7]. Among them, we can pick up points belonging to lower left boundary segments.

The above discussion indicates that we need to know the points $(0, m)$ and (a_L, b_L) before using convex hull algorithms to determine the lower left boundary segments. In the rest of this section, we present a method to find (a_L, b_L) numerically.

First, we present a property of b_L.

LEMMA 3.9. $b_L = m_a$ *with m_a be the number of neutral chains whose poles are given as $s_n = \lambda_n/\tau$, where $n \in \mathbb{Z}$, $n \to \infty$ and λ_n are defined as in* (5).

Proof. $m - m_a$ is the total number of non-zero values of ν_1. This number is then equal to $(\max\{b \mid (a, b) \in \cup_{\mathcal{S} \in \mathfrak{S}(AB(r))}\mathcal{S}\} - \min\{b \mid (a, b) \in \cup_{\mathcal{S} \in \mathfrak{S}(AB(r))}\mathcal{S}\})$ since the number of non-zero values of ν_1 for each $\mathcal{S} \in \mathfrak{S}(AB(r))$ is $(\max\{b \mid (a, b) \in \mathcal{S}\} - \min\{b \mid (a, b) \in \mathcal{S}\})$ and the segments in $\mathfrak{S}(AB(r))$ are interconnected. Also note that $\max\{b \mid (a, b) \in \cup_{\mathcal{S} \in \mathfrak{S}(AB(r))}\mathcal{S}\} = m$ and $\min\{b \mid (a, b) \in \cup_{\mathcal{S} \in \mathfrak{S}(AB(r))}\mathcal{S}\} = b_L$. □

The next lemma provides a tool to derive the number of chains of poles with $s_n = \lambda_n/\tau$.

LEMMA 3.10. *Let $G(s)$ be a neutral delay system defined by* (1) *and $d(s)$ be its denominator, i.e.*

$$d(s) := p(s) + \sum_{k=1}^{N} q_k(s)e^{-ks\tau}.$$

Let us denote by r a root of multiplicity m of $\widetilde{c}_d(z)$ defined by (3). *The following statements are equivalent:*

(i) $d(s)$ has m_a identical chains of poles s_n on the asymptotic axis corresponding to r with $s_n = \lambda_n/\tau$ where $n \in \mathbb{Z}$, n is large enough and λ_n is given by (5).

(ii)

$$(11) \qquad \left.\frac{\mathrm{d}^b d(s)}{\mathrm{d}s^b}\right|_{e^{-s\tau}=r} \equiv 0, \quad b = 0, \ldots, m_a - 1,$$

$$(12) \qquad \left.\frac{\mathrm{d}^{m_a} d(s)}{\mathrm{d}s^{m_a}}\right|_{e^{-s\tau}=r} \not\equiv 0,$$

where $\frac{\mathrm{d}^0 d(s)}{\mathrm{d}s^0} = d(s)$ and by $\left.\frac{\mathrm{d}^b d(s)}{\mathrm{d}s^b}\right|_{e^{-s\tau}=r}$ for $b = 1, \ldots, m_a$ we mean to substitute the exponential term $e^{-s\tau}$ by r after taking the b-th derivative of $d(s)$.

Proof. (i) $\implies$ (ii):
The fact that $s_n = \lambda_n/\tau$ for $n \in \mathbb{Z}$ and n large enough are roots of multiplicity

m_a of $d(s)$ is equivalent to

$$\frac{\mathrm{d}^b d(s_n)}{\mathrm{d} s^b} = 0, \quad b = 0, \ldots, m_a - 1, \tag{13}$$

$$\frac{\mathrm{d}^{m_a} d(s_n)}{\mathrm{d} s^{m_a}} \neq 0. \tag{14}$$

Since (13) and the equality $e^{-s_n \tau} = r$ hold for an infinite number of $n \in \mathbb{Z}$, then $\left.\frac{\mathrm{d}^b d(s)}{\mathrm{d} s^b}\right|_{e^{-s\tau}=r} = 0$, $b = 0, \ldots, m_a - 1$ have infinitely many roots s_n, $n \in \mathbb{Z}$. However, as $\left.\frac{\mathrm{d}^b d(s)}{\mathrm{d} s^b}\right|_{e^{-s\tau}=r}$ are polynomials in s^μ, $\mu \in (0, 1]$, they only have a finite number of roots and so necessarily (11) is satisfied.

It is obvious that (14) implies (12).

(ii) $\implies$ (i):

From (11), we deduce that $s_n = \lambda_n / \tau$, $n \in \mathbb{Z}$ are roots of $\left.\frac{\mathrm{d}^b d(s)}{\mathrm{d} s^b}\right|_{e^{-s\tau}=r}$, $b = 0, \ldots, m_a - 1$. Furthermore, s_n are roots of $e^{-s\tau} = r$. Therefore, s_n are roots of $\frac{\mathrm{d}^b d(s)}{\mathrm{d} s^b}$.

On the other hand, due to (12), the polynomial $\left.\frac{\mathrm{d}^{m_a} d(s)}{\mathrm{d} s^{m_a}}\right|_{e^{-s\tau}=r}$ has a finite number of roots and its roots are bounded. Therefore, there exists $N_1 \in \mathbb{Z}_+$ such that for $|n| > N_1$, s_n are not roots of $\left.\frac{\mathrm{d}^{m_a} d(s)}{\mathrm{d} s^{m_a}}\right|_{e^{-s\tau}=r}$ and thus are not roots of $\frac{\mathrm{d}^{m_a} d(s)}{\mathrm{d} s^{m_a}}$.

Hence, we conclude that s_n with $n \in \mathbb{Z}$ and n large enough are roots of multiplicity m_a of $d(s)$. $\qquad \square$

The previous lemma shows that by checking $\left.\frac{\mathrm{d}^b d(s)}{\mathrm{d} s^b}\right|_{e^{-s\tau}=r}$ starting from $b = 0$, we can determine $b_L = m_a$ which is the smallest value of b such that the derivative is not identically zero. After determining b_L, we can determine a_L by running a loop to find the smallest value of a such that $\sum_{k=1}^N \alpha_{a,k} k^{b_L} r^k \neq 0$.

4. Stability. In this section, we study whether or not a system is H_∞-stable based on the approximation of poles obtained in the preceding section. Here, we are only interested in systems with neutral chains asymptotic to the imaginary axis.

The next theorem provides quick tests on the instability of the systems. It does not even require to know $\nu_{n,1}$.

THEOREM 4.1. *Let $G(s)$ be a neutral delay system defined by (1), and suppose that the formal polynomial $\widetilde{c}_d(z)$ defined in (3) has roots of modulus one. If for such a root, denoted by r, there exists $\mathcal{S} \in \mathfrak{S}(AB(r))$ with $AB(r)$ defined in (6) such that $n(\mathcal{S}) = 2$ and either of the following conditions holds for $(a_1, b_1), (a_2, b_2) \in \mathcal{S}, b_1 > b_2$*

- *$b_1 - b_2 \geq 3$,*
- *$b_1 - b_2 = 2$, and $(a_2 - a_1)\mu \neq 2k, k \in \mathbb{Z}_+ \backslash \{0\}$,*

then the system is unstable[1].

Proof. The value of ν_1 as defined by Theorem 3.4 are given by

$$\nu_1^{b_1 - b_2} = -\frac{\tau^{(a_2 - a_1)\mu}}{(j2\pi)^{(a_2 - a_1)\mu}} \frac{(-1)^{(b_2 - b_1)} b_1!}{b_2!} \frac{\sum_{k=1}^N \alpha_{a_2,k} k^{b_2} r^k}{\sum_{k=1}^N \alpha_{a_1,k} k^{b_1} r^k}.$$

From now on we only consider positive n since poles of $G(s)$ are symmetric w.r.t. the real axis. With positive n, a ν_1 with positive real part implies a chain of poles on the right of the imaginary axis.

---

[1] The system is unstable in the sense that it is not H_∞-stable.

It is easy to see that for $b_1 - b_2 \geq 3$ there exists at least one value of ν_1 with positive real part.

Now consider the case where $b_1 - b_2 = 2$ and $(a_2 - a_1)\mu \neq 2k, k \in \mathbb{Z}_+\backslash\{0\}$. Let us denote

$$K_r = \frac{\sum_{k=1}^{N} \alpha_{a_2,k} k^{b_2} r^k}{\sum_{k=1}^{N} \alpha_{a_1,k} k^{b_1} r^k}.$$

If $r \in \mathbb{R}$, then $K_r \in \mathbb{R}$. However, since $(a_2 - a_1)\mu \neq 2k, k \in \mathbb{Z}_+\backslash\{0\}$ then $\jmath^{(a_2-a_1)\mu} = e^{\jmath\frac{(a_2-a_1)\mu\pi}{2}} \in \mathbb{C}\backslash\mathbb{R}$, thus leading to $\nu_1^2 \in \mathbb{C}\backslash\mathbb{R}$. This indicates that the two values of ν_1 have non-zero real parts. Since they are symmetric w.r.t. the origin then one of them has positive real part, which implies that the system is unstable.

If $r \in \mathbb{C}\backslash\mathbb{R}$, then $\bar{r}$ is also a root of the polynomial $\widetilde{c}_d(z)$ given as in (3). Denote $\nu_{1(r)}$ and $\nu_{1(\bar{r})}$ the values of ν_1 relative to r and $\bar{r}$ respectively. We obtain thus

$$\nu_{1(r)}^2 + \nu_{1(\bar{r})}^2 = -\frac{\tau^{(a_2-a_1)\mu}}{(\jmath 2\pi)^{(a_2-a_1)\mu}} \frac{(-1)^{(b_2-b_1)} b_1!}{b_2!} 2\Re(K_r),$$

which is not real. It turns out that either $\nu_{1(r)}^2$ or $\nu_{1(\bar{r})}^2$ is not real, thus giving at least one value of ν_1 with positive real part. $\square$

In the favorable case where neutral chains approach the imaginary axis from the left, the next theorem presents conditions for the system to be H_∞-stable.

To facilitate the proof of the theorem, we first state a primary result.

LEMMA 4.2. *Let $AB(r)$ be a set in $\mathbb{R}^2$ defined as in (6). Suppose that $\mathfrak{S}(AB(r)) \neq \emptyset$. Let $\mathcal{S}_L \in \mathfrak{S}(AB(r))$ be the segment containing (a_L, b_L). Then for all $\mathcal{S} \in \mathfrak{S}(AB(r))$, every point $(a, b) \in \mathcal{S}$ satisfies $a + b\mathfrak{p}(\mathcal{S}) \leq a_L + b_L\mathfrak{p}(\mathcal{S}_L)$.*

Proof. Let $\mathcal{S} \in \mathfrak{S}(AB(r))$. We consider $(a, b) \in \mathcal{S}$ and $(a, b) \neq (a_L, b_L)$.

By definition, $a + b\mathfrak{p}(\mathcal{S}) \leq a_L + b_L\mathfrak{p}(\mathcal{S})$, which leads to $\mathfrak{p}(\mathcal{S}) \leq (a_L - a)/(b - b_L)$ since $b > b_L$.

Also by definition, $a_L + b_L\mathfrak{p}(\mathcal{S}_L) \leq a + b\mathfrak{p}(\mathcal{S}_L)$, which leads to $\mathfrak{p}(\mathcal{S}_L) \geq (a_L - a)/(b - b_L)$.

Therefore, $\mathfrak{p}(\mathcal{S}) \leq \mathfrak{p}(\mathcal{S}_L)$, and thus $a + b\mathfrak{p}(\mathcal{S}) \leq a_L + b_L\mathfrak{p}(\mathcal{S}) \leq a_L + b_L\mathfrak{p}(\mathcal{S}_L)$. $\square$

THEOREM 4.3. *Let $G(s)$ be a neutral delay system defined by (1), and suppose that G has no unstable poles of small modulus as well as no unstable chains of poles. Also suppose that the formal polynomial $\widetilde{c}_d(z)$ defined in (3) has roots of modulus one, denoted by r, and that all values of ν_1 relative to each r satisfy $\Re(\nu_1) < 0$ where ν_1 is defined by (7). Then G is H_∞-stable if and only if $\deg p \geq \deg t + \max_r\{a_L\}$ with (a_L, b_L) defined as in (10) (it is the leftmost point among the lowest points of $AB(r)$).*

Proof. Since G has poles approaching the imaginary axis, then $|G(s)|_{s\in\jmath\mathbb{R}}$ is large near these asymptotic poles.

Let us consider the denominator of G at a point s on the imaginary axis near an asymptotic pole relative to a root r of modulus one of $\widetilde{c}_d(z)$. We can write $s = s_n + \eta_n \in \jmath\mathbb{R}$, where s_n is one of the poles of the neutral chain relative to r. Recall that $s_n = (\lambda_n + \nu_1 n^{-y_1})/\tau + o(n^{-y_1})$. Since $\Re(\nu_1) \neq 0$, then η_n is at least of order $(-y_1)$ and has the form $\eta_n = \eta n^{-y_1} + o(n^{-y_1})$. We can then write

$$(15) \qquad s = \frac{\lambda_n}{\tau} + \frac{\nu_1 + \eta\tau}{\tau n^{y_1}} + o(n^{-y_1}).$$

Note that s is of the same form as s_n if we denote $\nu_1' = \nu_1 + \eta\tau$.

Therefore, the developments of the denominator of G around s as $|s| \to \infty$ and around s_n as $|s_n| \to \infty$ are the same. Recall from (9) and the discussion that follows the equation that the development of $d(s_n)$ as $|s_n| \to \infty$ is

$$d(s_n) = p(s_n) \left(\frac{f_i(\nu_1)}{n^{(a+b\mathfrak{p}(\mathcal{S}))\mu}} + o(n^{-(a+b\mathfrak{p}(\mathcal{S}))\mu}) \right),$$

where $(a,b) \in \mathcal{S}$ and $f_i(\nu_1) = \sum_{(a,b) \in \mathcal{S}} C(a,b,\nu_1)$ for each $\mathcal{S} \in \mathfrak{S}(AB(r))$. Hence, the development of $d(s)$ for $s \in \jmath\mathbb{R}$ near s_n is

$$d(s) = p(s) \left(\frac{f_i(\nu_1 + \eta\tau)}{n^{(a+b\mathfrak{p}(\mathcal{S}))\mu}} + o(n^{-(a+b\mathfrak{p}(\mathcal{S}))\mu}) \right).$$

Since $s \in \jmath\mathbb{R}$, then (15) shows that $\Re(\nu_1 + \eta\tau) = 0$, and thus $f_i(\nu_1 + \eta\tau) = f_i(\jmath\Im(\nu_1 + \eta\tau))$. Since every root of $f_i(\nu_1)$ has strictly negative real part by assumption, then $f_i(\jmath\Im(\nu_1 + \eta\tau)) \neq 0$. Hence, the order of the denominator of $G(s)$ is $n^{(d_0-a-b\mathfrak{p}(\mathcal{S}))\mu}$ where $d_0 = \deg p$.

The assumption that G has no unstable poles implies that G has no chains of poles on the imaginary axis, and thus the leftmost lowest point of $AB(r)$ is $(a_L, 0)$. Due to Lemma 4.2, $a + b\mathfrak{p}(\mathcal{S}) \leq a_L$ for all $\mathcal{S} \in \mathfrak{S}(AB(r))$. Then the lowest order of the denominator of $G(s)$ for $s \in \jmath\mathbb{R}$ near s_n relative to r is $n^{(d_0-a_L)\mu}$.

For all roots r of $\widetilde{c}_d(z)$, the lowest order of the denominator of $G(s)$ on the imaginary axis is $n^{(d_0-x)\mu}$ with $x = \max_r\{a_L\}$. $\qquad\square$

5. Examples of different classes of systems.
In subsection 5.1 we will revisit two classes of neutral systems which have been presented in [17, 16]: one is standard, the other is fractional. Even though being similar, they have been separately studied. Here, both systems will be examined at the same time with the use of the method proposed in this paper. The obtained theoretical results on pole approximation, which indicates pole location w.r.t. the asymptotic axis, and stability conditions will be shown to be the same as before. Application of the unified method to other classes of systems considered in [4, 18, 16, 17] can be found in [15].

In subsection 5.2 we will investigate a new class of standard and fractional systems using the present approach. We will provide a numerical example with a full analysis: position of the chains of poles and test of H_∞-stability.

5.1. Systems with $m \geq 2$, $\sum_{k=1}^{N} \alpha_{1,k} r^k = 0$, $\sum_{k=1}^{N} \alpha_{1,k} k r^k \neq 0$, and $\sum_{k=1}^{N} \alpha_{2,k} r^k \neq 0$.
In this subsection, we consider (fractional) neutral systems $G(s)$ given by (1). In addition, its formal polynomial has a root r with multiplicity $m \geq 2$ satisfying the conditions

$$\sum_{k=1}^{N} \alpha_{1,k} r^k = 0, \qquad \sum_{k=1}^{N} \alpha_{1,k} k r^k \neq 0, \qquad \sum_{k=1}^{N} \alpha_{2,k} r^k \neq 0.$$

Classical and fractional neutral systems of this type were studied in [17, Subsection 5.2] and in [16, Subsection III.A] respectively. Recall that in this paper we make some changes of notation. In comparison with [16, 17], $\alpha_{0,k} = \alpha_k$, $\alpha_{1,k} = \beta_k$, $\alpha_{2,k} = \gamma_k$, and $\alpha_{3,k} = \delta_k$.

Since $\sum_{k=1}^{N} \alpha_{1,k} r^k = 0$, $\sum_{k=1}^{N} \alpha_{1,k} k r^k \neq 0$, and $\sum_{k=1}^{N} \alpha_{2,k} r^k \neq 0$, then $(1,0) \notin AB(r)$ and $(1,1),(2,0) \in AB(r)$. Recall from our discussion after Theorem 3.4 that $(0,m) \in AB(r)$.

5.1.1. The case where $m = 2$. It is easy to see that $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1\}$ with $\mathcal{S}_1 = \{(0,2),(1,1),(2,0)\}$ (see Figure 3). Therefore, from Theorem 3.4 we obtain

$$\sum_{(a,b)\in\mathcal{S}_1} C(a,b,\nu_1) = \frac{\nu_1^2}{2}\sum_{k=1}^N \alpha_{0,k}k^2 r^k - \frac{\tau^\mu}{(j2\pi)^\mu}\nu_1\sum_{k=1}^N \alpha_{1,k}kr^k + \frac{\tau^{2\mu}}{(j2\pi)^{2\mu}}\sum_{k=1}^N \alpha_{2,k}r^k = 0$$

and $y_1 = \mathfrak{p}(\mathcal{S}_1)\mu = \mu$. Identical results were presented in [17, Theorem 5.3] and [16, Theorem 1].

For fractional systems which have no unstable poles but have neutral chains of poles approaching the imaginary axis, if all these chains are relative to double roots of the formal polynomial $\widetilde{c}_d(z)$ that satisfy the conditions in this subsection, then from Theorem 4.3 these systems are stable in the sense H_∞ if and only if $\deg t \leq \deg p - \max_r\{a_L\}$. Here, $\max_r\{a_L\} = 2$ since the leftmost lowest point of $AB(r)$ is $(a_L, b_L) = (2,0)$ for all r being a root of modulus one of $\widetilde{c}_d(z)$. The same stability condition was obtained in [17, Proposition 5.5]. An example of these systems can be found in [17, Example 2].

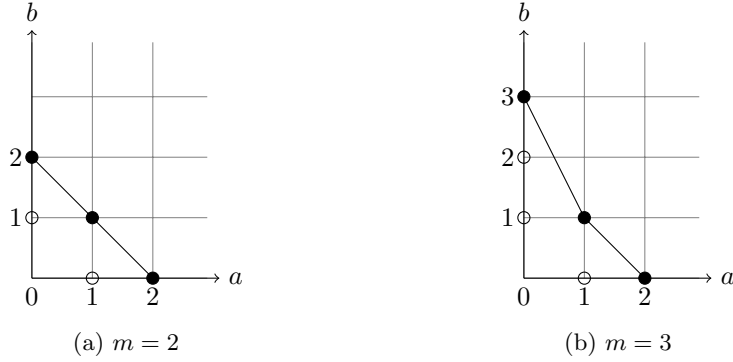

(a) $m = 2$ (b) $m = 3$

Fig. 3: The lower left boundary segments of $AB(r)$ in the case where $m \geq 2$, $\sum_{k=1}^N \alpha_{1,k}r^k = 0$, $\sum_{k=1}^N \alpha_{1,k}kr^k \neq 0$, and $\sum_{k=1}^N \alpha_{2,k}r^k \neq 0$. The black and white dots represent respectively points in and not in $AB(r)$.

5.1.2. The case where $m \geq 3$. We have $\mathfrak{S}(AB(r)) = \{\mathcal{S}_1, \mathcal{S}_2\}$ with $\mathcal{S}_1 = \{(0,m),(1,1)\}$ and $\mathcal{S}_2 = \{(1,1),(2,0)\}$ (see Figure 3). Therefore,

$$\sum_{(a,b)\in\mathcal{S}_1} C(a,b,\nu_1) = 0 \text{ and } y_1 = \mathfrak{p}(\mathcal{S}_1)\mu,$$

$$\sum_{(a,b)\in\mathcal{S}_2} C(a,b,\nu_1) = 0 \text{ and } y_1 = \mathfrak{p}(\mathcal{S}_2)\mu,$$

which are respectively equivalent to

$$\nu_1^{m-1} = \frac{(-1)^m m! \tau^\mu \sum_{k=1}^N \alpha_{1,k}kr^k}{(j2\pi)^\mu \sum_{k=1}^N \alpha_{0,k}k^m r^k} \text{ and } y_1 = \frac{\mu}{m-1},$$

$$\nu_1 = \frac{\tau^\mu \sum_{k=1}^N \alpha_{2,k}r^k}{(j2\pi)^\mu \sum_{k=1}^N \alpha_{1,k}kr^k} \text{ and } y_1 = \mu.$$

These results are the same as those shown in [17, Theorem 5.3] and [16, Theorem 1].

If $m = 3$, we have $n(\mathcal{S}_1) = 2$, $b_1 - b_2 = m - 1 = 2$ and $(a_2 - a_1)\mu = (1 - 0)\mu = \mu \neq 2k, k \in \mathbb{Z}_+\backslash\{0\}$, then according to Theorem 4.1 the system is unstable.

If $m \geq 4$, we have $n(\mathcal{S}_1) = 2$ and $b_1 - b_2 = m - 1 \geq 3$, then the system is unstable.

In [17, Corollary 5.4] and [16, Corollary 1], we derived the same conclusions about the stability of the system.

5.2. Systems with $m \geq 2$, $\sum_{k=1}^{N} \alpha_{1,k}r^k = 0$, $\sum_{k=1}^{N} k\alpha_{1,k}r^k = 0$, $\sum_{k=1}^{N} k^2\alpha_{1,k}r^k \neq 0$, $\sum_{k=1}^{N} \alpha_{2,k}r^k = 0$, $\sum_{k=1}^{N} k\alpha_{2,k}r^k \neq 0$, **and** $\sum_{k=1}^{N} \alpha_{3,k}r^k \neq 0$. This case has not been covered by [4, 18, 16, 17]. Instead of repeating all the calculations and analyses step-by-step as in these works for this new class of systems, we derive rapidly the results with the present method.

The most important task of finding the lower left boundary segments, which can be done by convex hull algorithms as discussed in subsection 3.2, is obvious in this case as it is for the previous case. From the conditions, we deduce that $(1,0), (1,1), (2,0) \notin AB(r)$ and $(1,2), (2,1), (3,0) \in AB(r)$. We also have $(0,m) \in AB(r)$. Hence, the lower left boundary segments for different m are given as follows

- for $m = 2$, $\mathcal{S}_1 = \{(0,2), (3,0)\}$;
- for $m = 3$, $\mathcal{S}_1 = \{(0,3), (1,2), (2,1), (3,0)\}$;
- for $m \geq 4$, $\mathcal{S}_1 = \{(0,m), (1,2)\}$ and $\mathcal{S}_2 = \{(1,2), (2,1), (3,0)\}$.

They are illustrated in Figure 4.

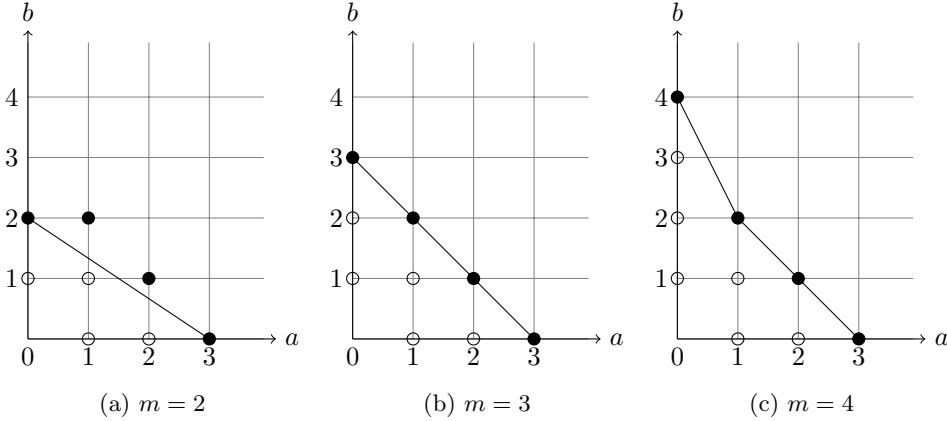(a) $m = 2$ (b) $m = 3$ (c) $m = 4$

Fig. 4: The lower left boundary segments of $AB(r)$ in the case where $m \geq 2$, $\sum_{k=1}^{N} \alpha_{1,k}r^k = 0$, $\sum_{k=1}^{N} k\alpha_{1,k}r^k = 0$, $\sum_{k=1}^{N} k^2\alpha_{1,k}r^k \neq 0$, $\sum_{k=1}^{N} \alpha_{2,k}r^k = 0$, $\sum_{k=1}^{N} k\alpha_{2,k}r^k \neq 0$, and $\sum_{k=1}^{N} \alpha_{3,k}r^k \neq 0$

From Theorem 3.4, we obtain the first approximation term $\nu_{n,1} = \nu_1/n^{y_1}$ corresponding to the segments, which are summarized in Table 1.

Even without calculating $\nu_{n,1}$, we can conclude, using Theorem 4.1, that the systems with r of multiplicity $m = 2$ are unstable if $\mu \neq 2/3$, and those with r of multiplicity $m \geq 4$ are unstable for all $\mu \in (0,1]$.

In the case $m = 3$, for fractional systems which have no unstable poles and whose chains of poles asymptotic to the imaginary axis correspond to the triple roots of the formal polynomial $\widetilde{c}_d(z)$ that satisfy the conditions in this subsection, Theorem 4.3

m	ν_1	y_1
2	$\nu_1^2 = -\dfrac{2\tau^{3\mu}\sum_{k=1}^{N}\alpha_{3,k}r^k}{(\jmath 2\pi)^{3\mu}\sum_{k=1}^{N}\alpha_{0,k}k^2r^k}$	$y_1 = \dfrac{3\mu}{2}$
3	$-\dfrac{\nu_1^3}{3!}\sum_{k=1}^{N}\alpha_{0,k}k^3r^k + \dfrac{\nu_1^2\tau^{\mu}}{2(\jmath 2\pi)^{\mu}}\sum_{k=1}^{N}\alpha_{1,k}k^2r^k$ $-\dfrac{\nu_1\tau^{2\mu}}{(\jmath 2\pi)^{2\mu}}\sum_{k=1}^{N}\alpha_{2,k}kr^k + \dfrac{\tau^{3\mu}}{(\jmath 2\pi)^{3\mu}}\sum_{k=1}^{N}\alpha_{3,k}r^k = 0$	$y_1 = \mu$
≥ 4	$\nu_1^{m-2} = \dfrac{(-1)^{m+1}m!\tau^{\mu}\sum_{k=1}^{N}\alpha_{1,k}k^2r^k}{2(\jmath 2\pi)^{\mu}\sum_{k=1}^{N}\alpha_{0,k}k^mr^k}$	$y_1 = \dfrac{\mu}{m-2}$
	$\dfrac{\nu_1^2\tau^{\mu}}{2(\jmath 2\pi)^{\mu}}\sum_{k=1}^{N}\alpha_{1,k}k^2r^k - \dfrac{\nu_1\tau^{2\mu}}{(\jmath 2\pi)^{2\mu}}\sum_{k=1}^{N}\alpha_{2,k}kr^k$ $+\dfrac{\tau^{3\mu}}{(\jmath 2\pi)^{3\mu}}\sum_{k=1}^{N}\alpha_{3,k}r^k = 0$	$y_1 = \mu$

Table 1: The values of (ν_1, y_1) in the case where $m \geq 2$, $\sum_{k=1}^{N}\alpha_{1,k}r^k = 0$, $\sum_{k=1}^{N}k\alpha_{1,k}r^k = 0$, $\sum_{k=1}^{N}k^2\alpha_{1,k}r^k \neq 0$, $\sum_{k=1}^{N}\alpha_{2,k}r^k = 0$, $\sum_{k=1}^{N}k\alpha_{2,k}r^k \neq 0$, and $\sum_{k=1}^{N}\alpha_{3,k}r^k \neq 0$

can be applied. It shows that the necessary and sufficient for the H_∞-stability of the systems is $\deg t \leq \deg p - \max_r\{a_L\}$ where $\max_r\{a_L\} = 3$ since $a_L = 3$ for all the roots of modulus one of $\widetilde{c}_d(z)$. Such a fractional system is presented in Example 5.1.

Example 5.1. The system is described by the transfer function

$$G(s) = \frac{t(s)}{d(s)},$$

where the characteristic equation of the system is a product of the characteristic equations of 3 single time-delay systems and is given by

$$\begin{aligned}
d(s) &= [(s^{0.2}+1)+s^{0.2}e^{-s}][(s^{0.2}+2)+(s^{0.2}-1)e^{-s}][(s^{0.2}+3)+(s^{0.2}+1)e^{-s}] \\
&= s^{0.6}+6s^{0.4}+11s^{0.2}+6+(3s^{0.6}+12s^{0.4}+10s^{0.2}-1)e^{-s} \\
&\quad + (3s^{0.6}+6s^{0.4}-2s^{0.2}-1)e^{-2s}+(s^{0.6}-s^{0.2})e^{-3s}.
\end{aligned}$$

The formal polynomial of this system is $\widetilde{c}_d(z) = 1 + 3z + 3z^2 + z^3$ and has $r = -1$ of multiplicity $m = 3$. There are then 3 neutral chains of poles approaching the imaginary axis. The conditions in this subsection are all satisfied. Therefore, $\nu_{n,1} = \nu_1 n^{-0.2}$ where ν_1 is given in Table 1 (the case $m = 3$) and has three values $-0.6585+0.2140\jmath$, $-1.3170+0.4279\jmath$, and $-1.9756+0.6419\jmath$. The upper parts of the chains of poles are then on the left of the imaginary axis and so are the lower parts since poles are symmetric about the real axis.

We obtain the same values of $\nu_{n,1}$ if considering separately each factor of the characteristic equation using the results in [17, Theorem 4.1].

The poles of small modulus are also in the open left half-plane as we can see in Figure 5. Therefore, according to Theorem 4.3, we derive, as in the discussion before this example, that the system is H_∞-stable if and only if $\deg t \leq \deg p - 3$.

Since $\deg p = 3$, then the only choice is $\deg t = 1$, i.e. $t(s)$ is a constant. Figures 6 and 7 show the magnitude of the transfer function when $t(s) = 1$ and $t(s) = s^{0.2} + 2$ respectively. The transfer function is bounded in the former case and unbounded in the latter one.
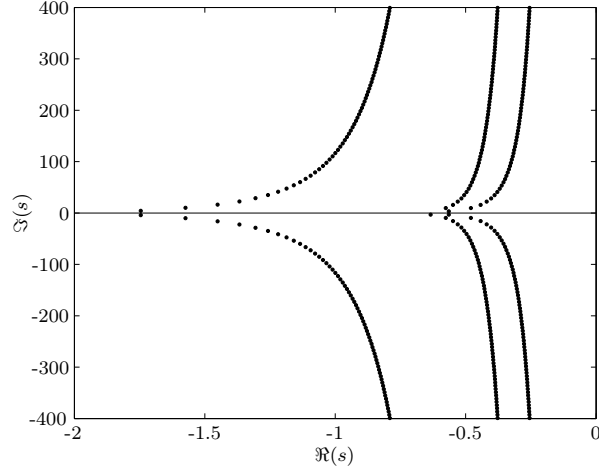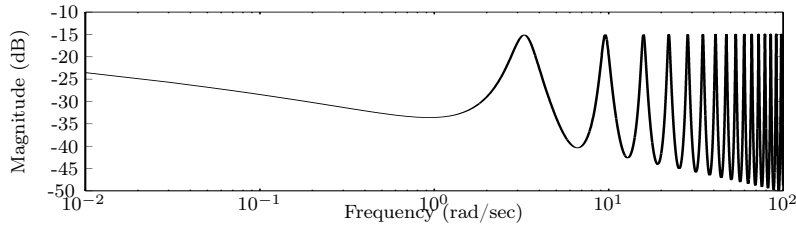

Fig. 5: Poles of $G(s)$

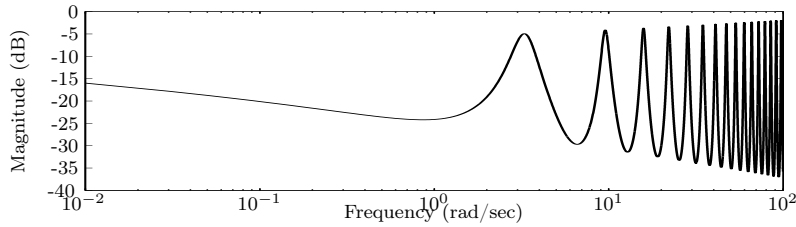Fig. 6: Bode diagram of $G(s)$ with $t(s) = 1$

Fig. 7: Bode diagram of $G(s)$ with $t(s) = s^{0.2} + 2$

6. Conclusion. In this paper we have considered the H_∞-stability of standard and fractional neutral systems with commensurate delays and chains of poles asymptotic to the imaginary axis. More precisely, we have studied the location of these

chains of poles around the axis and the boundedness of the transfer function on the axis. The new results generalize those presented in [4, 18, 16, 17]. They concern both classical and fractional systems and cover all possible cases, some of which were studied separately in the previous papers.

Being general, the new results allow concise programming in contrast to the tedious case-by-case implementation of the previous results. Their programming is even easier by the use of available convex hull algorithms to obtain lower left boundary segments, which are of central importance in this paper. The implementation of the present approach in the Matlab toolbox YALTA [2] to facilitate its use is under study.

Appendix A. Proof of Lemma 3.5.
We have

$$\frac{d(s_n)}{p(s_n)} = 1 + \sum_{k=1}^{N} \frac{q_k(s_n)}{p(s_n)} e^{-ks_n\tau}.$$

As $|s_n| \to \infty$, using (2) leads to

$$\frac{d(s_n)}{p(s_n)} = 1 + \sum_{k=1}^{N} \left(\alpha_{0,k} + \sum_{l=1}^{M'} \frac{\alpha_{l,k}}{s_n^{l\mu}} + \mathcal{O}(s_n^{-(M'+1)\mu}) \right) e^{-ks_n\tau}.$$

Replace s_n with the expression (8) and note that

$$e^{-\lambda_n} = r,$$

and

$$e^{-k\nu_{n,i}} = 1 + \sum_{l=1}^{\left[\frac{M'\mu}{y_i}\right]} \frac{(-1)^l \nu_i^l k^l}{l! n^{ly_i}} + o(n^{-M'\mu}) \text{ with } l \in \mathbb{Z}_+\backslash\{0\},$$

we have thus, when n is large enough,

$$\frac{d(s_n)}{p(s_n)} = 1 + \sum_{k=1}^{N} \left(\alpha_{0,k} + \sum_{l=1}^{M'} \frac{\alpha_{l,k}\tau^{l\mu}}{(j2\pi n)^{l\mu}} \left(1 + O(n^{-1})\right) + o(n^{-M'\mu}) \right) r^k$$

$$\times \prod_{i=1}^{M} \left(1 + \sum_{l=1}^{\left[\frac{M'\mu}{y_i}\right]} \frac{(-1)^l \nu_i^l k^l}{l! n^{ly_i}} + o(n^{-M'\mu}) \right)$$

and we obtain

$$\frac{d(s_n)}{p(s_n)} = 1 + \sum_{k=1}^{N} \left(\alpha_{0,k} + \sum_{l=1}^{M'} \frac{\alpha_{l,k}\tau^{l\mu}}{(j2\pi n)^{l\mu}} \left(1 + O(n^{-1})\right) + o(n^{-M'\mu}) \right) r^k$$

$$\times \left(1 + \sum_{(l_1,\ldots,l_M)\in\mathcal{L}(M'\mu)} \frac{(-1)^{\sum_{i=1}^{M} l_i} \left(\prod_{i=1}^{M} \nu_i^{l_i}\right) k^{\sum_{i=1}^{M} l_i}}{\left(\prod_{i=1}^{M} l_i!\right) n^{\sum_{i=1}^{M} l_i y_i}} + o(n^{-M'\mu}) \right).$$

By simple calculations, we obtain

$$
\frac{d(s_n)}{p(s_n)} = 1 + \sum_{k=1}^{N} r^k \left(\alpha_{0,k} + \sum_{l=1}^{M'} \frac{\alpha_{l,k}\tau^{l\mu}}{(j2\pi n)^{l\mu}} \left(1 + O(n^{-1})\right) \right.
$$

$$
+ \alpha_{0,k} \sum_{(l_1,\ldots,l_M)\in\mathcal{L}(M'\mu)} \frac{(-1)^{\sum_{i=1}^{M} l_i} \left(\prod_{i=1}^{M} \nu_i^{l_i}\right) k^{\sum_{i=1}^{M} l_i}}{\left(\prod_{i=1}^{M} l_i!\right) n^{\sum_{i=1}^{M} l_i y_i}}
$$

$$
+ \sum_{(l,l_1,\ldots,l_M)\in\mathcal{L}_0(M'\mu)} \frac{\alpha_{l,k}\tau^{l\mu}}{(j2\pi n)^{l\mu}} \left(1 + O(n^{-1})\right) \frac{(-1)^{\sum_{i=1}^{M} l_i} \left(\prod_{i=1}^{M} \nu_i^{l_i}\right) k^{\sum_{i=1}^{M} l_i}}{\left(\prod_{i=1}^{M} l_i!\right) n^{\sum_{i=1}^{M} l_i y_i}}
$$

$$
\left. + o(n^{-M'\mu}) \right),
$$

and then

$$
\frac{d(s_n)}{p(s_n)} = 1 + \sum_{k=1}^{N} \alpha_{0,k} r^k + \sum_{l=1}^{M'} \frac{\tau^{l\mu}}{(j2\pi n)^{l\mu}} \left(1 + O(n^{-1})\right) \sum_{k=1}^{N} \alpha_{l,k} r^k
$$

$$
+ \sum_{(l_1,\ldots,l_M)\in\mathcal{L}(M'\mu)} \frac{(-1)^{\sum_{i=1}^{M} l_i} \left(\prod_{i=1}^{M} \nu_i^{l_i}\right)}{\left(\prod_{i=1}^{M} l_i!\right) n^{\sum_{i=1}^{M} l_i y_i}} \sum_{k=1}^{N} \alpha_{0,k} r^k k^{\sum_{i=1}^{M} l_i}
$$

$$
+ \sum_{(l,l_1,\ldots,l_M)\in\mathcal{L}_0(M'\mu)} \frac{\tau^{l\mu}}{(j2\pi n)^{l\mu}} \left(1 + O(n^{-1})\right) \frac{(-1)^{\sum_{i=1}^{M} l_i} \left(\prod_{i=1}^{M} \nu_i^{l_i}\right)}{\left(\prod_{i=1}^{M} l_i!\right) n^{\sum_{i=1}^{M} l_i y_i}}
$$

$$
\times \sum_{k=1}^{N} \alpha_{l,k} r^k k^{\sum_{i=1}^{M} l_i} + o(n^{-M'\mu}).
$$

Note that $1 + \sum_{k=1}^{N} \alpha_{0,k} r^k = 0$ and that $d(s_n) = 0$ since s_n with $n \in \mathbb{Z}$ are poles of $G(s)$. The conclusion is then immediate.

REFERENCES

[1] A. B. ABUSAKSAKA AND J. R. PARTINGTON, *BIBO stability of some classes of delay systems and fractional systems*, Systems & Control Letters, 64 (2014), pp. 43 – 46.

[2] D. AVANESSOFF, A. R. FIORAVANTI, C. BONNET, AND L. H. V. NGUYEN, H_∞-*stability analysis of (fractional) delay systems of retarded and neutral type with the Matlab toolbox YALTA*, in Delay systems: From Theory to Numerics and Applications, T. Vyhlídal, J. F. Lafay, and R. Sipahi, eds., vol. 1 of Advances in Delays and Dynamics, Springer, 2014. (software available at http://gforge.inria.fr/projects/yalta-toolbox/).

[3] R. BELLMAN AND K. L. COOKE, *Differential-Difference Equations*, Academic Press, New York, London, 1 1963.

[4] C. BONNET, A. R. FIORAVANTI, AND J. R. PARTINGTON, *Stability of neutral systems with commensurate delays and poles asymptotic to the imaginary axis*, SIAM Journal on Control and Optimization, 49 (2011), pp. 498–516.

[5] C. BONNET AND J. R. PARTINGTON, *Analysis of fractional delay systems of retarded and neutral type*, Automatica, 38 (2002), pp. 1133–1138.

[6] C. Bonnet and J. R. Partington, *Stabilization of some fractional delay systems of neutral type*, Automatica, 43 (2007), pp. 2047–2053.

[7] M. De Berg, M. Van Kreveld, M. Overmars, and O. C. Schwarzkopf, *Computational Geometry*, Springer, 3rd ed., 2008.

[8] A. Elshurafa, M. N. Almadhoun, K. Salama, and H. Alshareef, *Microscale electrostatic fractional capacitors using reduced graphene oxide percolated polymer composites*, Applied Physics Letters, 102 (2013), p. 232901.

[9] T. Freeborn, *A survey of fractional-order circuit models for biology and biomedicine*, IEEE Journal on Emerging and Selected Topics in Circuits and Systems, 3 (2013), pp. 416–424.

[10] S. Hamamci, *An algorithm for stabilization of fractional-order time delay systems using fractional-order PID controllers*, IEEE Transactions on Automatic Control, 52 (2007), pp. 1964–1969.

[11] R. Hotzel, *Some stability conditions for fractional delay systems*, Journal of Mathematical Systems Estimation and Control, 8 (1998), pp. 1–19.

[12] A. Kilbas, H. Srivastava, and J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science, 1st ed., 2006.

[13] D. Matignon and B. d'Andréa Novel, *Spectral and time-domain consequences of an integro-differential perturbation of the wave PDE*, in 3rd International Conference on Mathematical and Numerical Aspects of Wave Propagation Phenomena, 1995, pp. 769–771.

[14] W. Michiels and S.-I. Niculescu, *Stability, Control, and Computation for Time-Delay Systems*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2014.

[15] L. H. V. Nguyen, *Stability and stabilization of several classes of fractional systems with delays*, PhD thesis, Université Paris Sud - Paris XI, December 2014.

[16] L. H. V. Nguyen and C. Bonnet, *H_∞-stability analysis of various classes of neutral systems with commensurate delays and with chains of poles approaching the imaginary axis*, in 54th IEEE Conference on Decision and Control, December 2015, pp. 6416–6421.

[17] L. H. V. Nguyen, C. Bonnet, and A. R. Fioravanti, *H_∞-stability analysis of fractional delay systems of neutral type*, SIAM Journal on Control and Optimization, 54 (2016), pp. 740–759.

[18] L. H. V. Nguyen, A. R. Fioravanti, and C. Bonnet, *Analysis of neutral systems with commensurate delays and many chains of poles asymptotic to same points on the imaginary axis*, in 10th IFAC Workshop on Time Delay Systems, June 2012, pp. 120–125.

[19] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, 1974.

[20] H. Özbay, C. Bonnet, and A. R. Fioravanti, *PID controller design for fractional-order systems with time delays*, Systems & Control Letters, 61 (2012), pp. 18–23.

[21] J. R. Partington, *Linear Operators and Linear Systems: An Analytical Approach to Control Theory*, Cambridge University Press, 2004.

[22] J. R. Partington and C. Bonnet, *H_∞ and BIBO stabilization of delay systems of neutral type*, Systems & Control Letters, 52 (2004), pp. 283–288.

[23] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and some of their Applications*, Academic Press, 1st ed., 1998.

[24] R. Rabah, G. M. Sklyar, and P. Y. Barkhayev, *Stability and stabilizability of mixed retarded-neutral type systems*, ESAIM: Control, Optimisation and Calculus of Variations, 18 (2012), pp. 656–692.

[25] J.-P. Richard, *Time-delay systems: an overview of some recent advances and open problems*, Automatica, 39 (2003), pp. 1667–1694.

[26] A. Si-Ammour, S. Djennoune, and M. Bettayeb, *A sliding mode control for linear fractional systems with input and state delays*, Communications in Nonlinear Science and Numerical Simulation, 14 (2009), pp. 2310–2318.

[27] T. Vyhlídal and P. Zitek, *QPmR v.2 – Quasipolynomial rootfinder, algorithm and examples*, in Delay systems: From Theory to Numerics and Applications, T. Vyhlídal, J. F. Lafay, and R. Sipahi, eds., vol. 1 of Advances in Delays and Dynamics, Springer, 2014. (software available at http://www.cak.fs.cvut.cz/algorithms/qpmr).

[28] S. Westerlund and L. Ekstam, *Capacitor theory*, IEEE Transactions on Dielectrics and Electrical Insulation, 1 (1994), pp. 826–839.