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Robust stability analysis of continuous PWA systems ⋆

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Abstract: This paper studies the robust stability of the origin of continuous-time Piecewise Affine (PWA) systems. We consider an implicit representation based on vector-valued ramp functions to model PWA continuous-time systems. From this representation, we derive Linear Matrix Inequality (LMI) stability conditions suitable to deal with polytopic uncertainties that can modify both the dynamics and the regions of the state-space partition of the PWA system. Two numerical examples illustrate the effectiveness of the proposed conditions.

Keywords: piecewise affine systems, uncertain system, stability analysis

1. INTRODUCTION

Piecewise Affine (PWA) systems are defined over partitions of the state-space, where each set of the partition is associated with an affine differential equation. Several practical systems can be modeled as PWA systems, such as nonlinear circuits (Rodrigues and Boyd, 2005), mechanical systems (Zou and Nagarajaiah, 2015) and systems with switches, relays, deadzones and saturations (Johansson, 2003; Heemels et al., 2001; Christophersen, 2007; Rodrigues and Boyd, 2005). Moreover, it is possible to approximate continuous nonlinearities in dynamic systems by PWA functions, leading to PWA models (Chen et al., 2021).

The most common representation for PWA systems is the explicit representation (Sontag, 1981)

$$\dot{x} = F_p x + f_p, \quad \forall x \in \Gamma_p, \tag{1a}$$

where x and $\dot{x} \in \mathbb{R}^n$ are, respectively, the state and its time derivative while the matrices $F_p \in \mathbb{R}^{n\times n}$ and vectors $f_p \in \mathbb{R}^n$ define the affine dynamic equation for each set Γ_p of the state-space partition, for $p = 1, \ldots, N_r$, where N_r is the number of sets. The sets Γ_p are commonly described as the intersection of half-spaces, defined by a finite number of inequalities, that is,

$$\Gamma_p = \{x \in \mathbb{R}^n \mid H_p x \succeq h_p\}, \tag{1b}$$

where $H_p \in \mathbb{R}^{q_p\times n}$ and $h_p \in \mathbb{R}^{q_p}$ define the regions Γ_p in terms of q_p hyperplanes, for $p = 1, \ldots, N_r$ (Nakada and Takaba, 2003). The partition can also be expressed in terms of cone rays and vertices (Iervolino et al., 2017). If the right hand side of (1a) is a continuous function, the system is called Continuous Piecewise Affine (CPWA).

Since PWA systems are nonlinear, multiple equilibria, limit cycles and chaotic trajectories can occur (Branicky, 1998; Utkin, 1977). Moreover, uncertainties may not only affect the dynamics within each set of the partition, but the partition itself, possibly changing the shape and the number of regions Γ_p. This motivates the development of methods to assess properties of the system such as convergence of trajectories to the origin in the presence of uncertainties.

However, only a few works propose methods to handle uncertain PWA system. For instance, in (Trimboli et al., 2011) a Piecewise Linear (PWL) Lyapunov function was considered to assess the stability of uncertain PWA systems. Furthermore, Hovd and Olaru (2018) propose a method relying in parameter-dependent PWQ Lyapunov functions. Both methods, however, are limited to the case where the partition, that is, the shape and number of regions, does not change due to uncertainties.

To overcome the intrinsic difficulties for the analysis using the explicit representation, such as the need of enumerating the regions, an implicit representation based on vector-valued ramp functions has been proposed for discrete-time PWA systems (Groff et al., 2019). A discussion on the equivalence of this implicit representation and the explicit one (1) is presented in (Cabral et al., 2021). This *implicit representation* was applied to the analysis of global stability (Groff et al., 2019, 2023) and global stabilization (Cabral et al., 2021) of the origin of discrete-time PWA systems.

The contributions of this work are therefore threefold. First, we propose the use of the implicit representation presented in (Groff et al., 2019) for the stability analysis of *continuous-time* PWA systems. Second, we propose a

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framework to analyse the robust stability of the origin of these systems considering polytopic uncertainties. Besides allowing to consider uncertainties on the dynamics in each region of the partition, this framework is also suitable for the case where both the shape of the partition and the number of regions may change, which is not possible with the methods of the aforementioned works. Finally, by using copositive matrices, we derive more relaxed conditions for the analysis of PWA systems in the implicit representation. This is an improvement over the previous results in (Groff et al., 2023) and (Cabral et al., 2021).

Notation: For a given vector $v \in \mathbb{R}^n$, $v_{(i)}$ represents its i-th element and $v \succeq 0$ denotes elementwise nonnegativity. For a given matrix $A \in \mathbb{R}^{n\times m}$, $A_{(i,j)}$ represents its element in the i-th row and j-th column. The block diagonal matrix formed by A and B is denoted by $\text{diag}(A, B)$. For square matrices of dimensions $n \times n$, $\text{He}(A) \triangleq A + A^\top$, $\mathcal{S}^n$ is the set of symmetric matrices, $\mathcal{S}^n_+$ is the set of elementwise nonnegative symmetric matrices, $\mathcal{C}^n$ is the set of symmetric copositive matrices (*i.e.*, $v^\top Av \geq 0\ \forall v \succeq 0$) and $\mathcal{D}^n$ is the set of diagonal matrices. The identity and the square zero matrices of size $n \times n$ are denoted as I_n and 0_n, while a non-square $n \times m$ matrix of zeros is given by $0_{n,m}$. Finally, $\mathbf{1}_n \in \mathbb{R}^n$ is a (column) vector of ones.

2. PROBLEM STATEMENT

Consider an uncertain continuous-time PWA in the implicit representation given by

$$\dot{x} = A(\theta)x + B(\theta)\phi(y(x)), \tag{2a}$$
$$y(x) = C(\theta)x + D(\theta)\phi(y(x)) + e(\theta), \tag{2b}$$

where $x \in \mathbb{R}^n$ is the state and $y \in \mathbb{R}^m$ is the argument of the vector-valued ramp function $\phi : \mathbb{R}^m \to \mathbb{R}^m$, which is defined elementwise in terms of the scalar ramp function $r : \mathbb{R} \to \mathbb{R}$ as

$$\phi_{(j)}(y) = r(y_{(j)}) = \begin{cases} y_{(j)} & \text{if } \ y_{(j)} > 0, \\ 0 & \text{if } \ y_{(j)} \leq 0, \ \ j = 1, \ldots, m, \end{cases}$$

while $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$ is the vector of uncertainties, belonging to the set of uncertainties Θ. Note that since the vector-valued ramp function ϕ is continuous, (2) is a CPWA system. The system dynamics is defined by the parameter dependent matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$ and the vector $e \in \mathbb{R}^m$.

Assumption 1. (Polytopic uncertainty). The dependency of $A(\theta)$, $B(\theta)$, $C(\theta)$, $D(\theta)$ and $e(\theta)$ can be expressed as

$$\begin{bmatrix} A(\theta) & B(\theta) & 0_n \\ C(\theta) & D(\theta) & e(\theta) \end{bmatrix} = \sum_{i=1}^{n_\theta} \theta_{(i)} \begin{bmatrix} A_i & B_i & 0_n \\ C_i & D_i & e_i \end{bmatrix}, \tag{3}$$

where A_i, B_i, C_i, D_i and e_i are given for $i = 1, \ldots, n_\theta$. Moreover, the set of uncertainties Θ is a unit simplex, that is,

$$\Theta \triangleq \left\{ \theta \in \mathbb{R}^{n_\theta} \ \middle| \ \sum_{i=1}^{n_\theta} \theta_{(i)} = 1, \ \ 0 \leq \theta_{(i)} \leq 1 \right\}. \tag{4}$$

In the case described by Assumption 1, the matrices A_i, B_i, C_i, D_i and the vectors e_i, for $i = 1, \ldots, n_\theta$, are called vertices of the uncertain system. Note that if the set of uncertainties Θ is bounded but not polytopic, it can be over-approximated by a polytopic set.

To verify the equivalence of the ramp-based implicit representation (2) and the explicit one (1), the following definition is useful.

Definition 1. (Active elements of ϕ). The j-th element of $\phi(y)$, $j = 1, \ldots, m$, for a given argument $y \in \mathbb{R}^m$ is said to be:

- *Active* if $\phi_{(j)}(y) > 0$, implying that $y_{(j)} > 0$;
- *Inactive* if $\phi_{(j)}(y) = 0$, implying that $y_{(j)} \leq 0$.

For a PWA system in the implicit representation (2), each set Γ_p of the state-space partition is encoded by the combination of *active* and *inactive* elements of $\phi(y(x))$ in the implicit equation (2b), according to Definition 1. Moreover, note that within a given set Γ_p the following relation is valid for a given θ:

$$\phi(y(x)) = \Delta_p y(x), \ \ x \in \Gamma_p,$$

where $\Delta_p \in \mathcal{D}^m$ with its elements defined as follows

$$\Delta_{p(j,j)} = \begin{cases} 1 & \text{if } \ \phi_{(j)}(y(x)) \text{ is active in } \Gamma_p, \\ 0 & \text{if } \ \phi_{(j)}(y(x)) \text{ is inactive in } \Gamma_p. \end{cases}$$

Thus, the hyperplanes that define each set Γ_p for a given θ in (1b) can be recovered with

$$\begin{aligned} H_p(\theta) &= \bar{\Delta}_p(I - D(\theta)\Delta_p)^{-1}C(\theta), \\ h_p(\theta) &= -\bar{\Delta}_p(I - D(\theta)\Delta_p)^{-1}e(\theta), \end{aligned} \tag{5}$$

where the j-th diagonal element of the matrix $\bar{\Delta}_p \triangleq 2\Delta_p - I_m$ is equal to 1 if $\phi_{(j)}(y(x))$ is active and -1 otherwise. Moreover, (1a) can be recovered with

$$\begin{aligned} F_p(\theta) &= A(\theta) + B(\theta)\Delta_p(I - D(\theta)\Delta_p)^{-1}C(\theta), \\ f_p(\theta) &= B(\theta)\Delta_p(I - D(\theta)\Delta_p)^{-1}e(\theta). \end{aligned} \tag{6}$$

Since some regions Γ_p can be empty for some $\theta \in \Theta$ in (1b) and (5), not only the dynamics within a region, but also the shape and the number of regions in (2) can be modified by the uncertain parameter θ. The following assumptions are considered in this work.

Assumption 2. (Equilibrium of the origin). The origin of the system in (2) is an equilibrium, that is, $B(\theta)\phi(y(0)) = 0$ is verified for any $\theta \in \Theta$.

Assumption 3. (Well-posedness). For any $\theta \in \Theta$ and $x \in \mathbb{R}^n$ the solution to the implicit equation (2b) exists and it is unique.

Note that Assumption 2 regarding the equilibrium of the origin of (2) is a necessary condition for the global stability of the origin. Furthermore, Assumption 3 is necessary to ensure that the trajectories are complete for any initial condition. Since from Assumption 1 the uncertainty is polytopic, Assumption 3 can be checked by verifying the well-posedness conditions presented in (Groff et al., 2023) only for the vertices of (3).

The problem of global exponential stability analysis of the uncertain system (2) can thus be stated as follows.

Problem 1. (Robust Stability Analysis). Given an uncertain CPWA (2)-(4), verify whether the trajectories converge exponentially to the origin for any initial condition $x_0 \in \mathbb{R}^n$ and any uncertainty $\theta \in \Theta$.

To address Problem 1 we will consider a quadratic Lyapunov candidate function $V : \mathbb{R}^n \to \mathbb{R}_+$ defined as

$$V(x) = x^\top Px, \tag{7}$$

where $P > 0 \in \mathcal{S}^n$. In the rest of the paper, sufficient LMI conditions are derived to verify whether (7) is a Lyapunov function to the uncertain system (2). With this aim, specific properties of ramp functions are used, as detailed in the next section.

3. PRELIMINARIES

This section states preliminaries results, which are instrumental to the robust stability analysis performed in Section 4.

The scalar ramp function r can be characterized by the following properties, which are valid for any scalar $a \in \mathbb{R}$ (Groff et al., 2019):

$$r(a) \geq 0, \tag{8a}$$
$$r(a) - a \geq 0, \tag{8b}$$
$$r(a)(r(a) - a) \equiv 0. \tag{8c}$$

Next, similar properties for the vector-valued ramp function ϕ will be provided based on (8). Let $\xi^\top(y(x)) \triangleq \begin{bmatrix}1 \; x^\top \; \phi^\top(y(x)) \; (\phi(y(x)) - y(x))^\top\end{bmatrix} \in \mathbb{R}^{n_\xi}$, where $n_\xi \triangleq 1 + n + 2m$. Then, the following lemmas can be stated.

Lemma 1. (Nonnegativity of ϕ). Let $M \in \mathcal{C}^{n_m}$, for $n_m \triangleq 1 + 2m$, and define

$$s_1(M, y(x)) \triangleq \xi^\top(y(x))\Pi_0^\top M \Pi_0 \xi(y(x)),$$

with

$$\Pi_0 = \begin{bmatrix} 1 & 0_{1,n} & 0_{1,2m} \\ 0_{2m,1} & 0_{2m,n} & I_{2m} \end{bmatrix}.$$

Then, the following inequality holds:

$$s_1(M, y(x)) \geq 0 \quad \forall x \in \mathbb{R}^n, \; \forall \theta \in \Theta. \tag{9}$$

Proof: Define

$$\chi^\top(y(x)) \triangleq \begin{bmatrix}1 \; \phi^\top(y(x)) \; (\phi(y(x)) - y(x))^\top\end{bmatrix}$$

and note that $\Pi_0\xi(y(x)) = \chi(y(x))$. From Assumption 3, $\exists y(x) \; \forall x \in \mathbb{R}^n$ and $\forall \theta \in \Theta$. Moreover, from (8a) and (8b), both the vector-valued ramp function $\phi(y(x))$ and its complement (*i.e.* $\phi(y(x)) - y(x)$) are nonnegative, yelding $\chi(y(x)) \succeq 0 \; \forall x \in \mathbb{R}^n$ and $\forall \theta \in \Theta$. Hence, if M is copositive, (9) holds. □

Lemma 2. (Complementarity of ϕ). Let $T \in \mathcal{D}^m$ and define

$$s_2(T, y(x)) \triangleq \xi^\top(y(x))\Pi_0^\top \Psi(T) \Pi_0 \xi(y(x))$$

with

$$\Psi(T) \triangleq \begin{bmatrix} 0 & 0_{1,m} & 0_{1,m} \\ 0_{m,1} & 0_m & T \\ 0_{m,1} & T & 0_m \end{bmatrix}.$$

Then, the following identity holds:

$$s_2(T, y(x)) \equiv 0 \quad \forall x \in \mathbb{R}^n, \; \forall \theta \in \Theta. \tag{10}$$

Proof: From Assumption 3, $\exists y(x) \; \forall x \in \mathbb{R}^n$ and $\forall \theta \in \Theta$. Since T is a diagonal matrix, $s_2(T, y(x))$ can be rewritten as

$$s_2(T, y(x)) = 2\sum_{j=1}^{m} T_{(j,j)}\phi_{(j)}(y(x))(\phi_{(j)}(y(x)) - y_{(j)}(x)).$$

From (8c), each term in the sum above is identically zero. Thus, (10) holds. □

Lemma 3. (Algebraic relation). Let $R \in \mathbb{R}^{n_\xi \times m}$ and define

$$s_3(R, y(x)) \triangleq \mathrm{He}(\xi^\top(y(x)) R Q(\theta) \xi(y(x))),$$

with

$$Q(\theta) \triangleq [e(\theta) \; C(\theta) \; (D(\theta) - I_m) \; I_m].$$

Then, the identity

$$s_3(R, y(x)) \equiv 0 \tag{11}$$

holds along the trajectories of the uncertain system (2).

Proof: From Assumption 3, $\exists y(x) \; \forall x \in \mathbb{R}^n$ and $\forall \theta \in \Theta$. Moreover, the identity $Q(\theta)\xi(y(x)) \equiv 0 \; \forall x \in \mathbb{R}^n$ and $\forall \theta \in \Theta$ follows from the algebraic relation (2b). Thus, (11) holds. □

In Lemma 1 we consider $M \in \mathcal{C}^{n_m}$, that is, a matrix M such that $v^\top M v \geq 0 \; \forall v \succeq 0$. In (Johansson, 1999, Chapter 8) the use of copositive matrices for the analysis of PWA systems using the explicit representation (1) was discussed. However, a subset of copositive matrices, namely the set of elementwise nonnegative matrices, is commonly used for this task. For instance, Mignone et al. (2000) and Feng (2002) considered elementwise nonnegative matrices for the stability results using the explicit representation (1) while Groff et al. (2019), Groff et al. (2023) and Cabral et al. (2021) consider this class of matrices in the results derived with the implicit representation (2). On the other hand, the following result from (Parrilo, 2000, Theorem 5.1) provides a less conservative condition to verify whether a matrix M is copositive.

Lemma 4. (Copositive matrices). Let $M \in \mathbb{R}^{n_m \times n_m}$. If there exist symmetric matrices $\Lambda_j \in \mathcal{S}^{n_m}$, for $j = 1, \ldots, n_m$, such that

$$\begin{aligned} M - \Lambda_j &\geq 0, \\ \Lambda_{j(j,j)} &= 0, \\ \Lambda_{j(k,k)} + \Lambda_{k(k,j)} + \Lambda_{k(j,k)} &= 0, \quad j \neq k, \\ \Lambda_{j(k,\ell)} + \Lambda_{k(\ell,j)} + \Lambda_{\ell(j,k)} &\geq 0, \quad j \neq k \neq \ell, \end{aligned} \tag{12}$$

then M is copositive, that is, $v^\top M v \geq 0 \; \forall v \succeq 0 \in \mathbb{R}^{n_m}$.

In the next section, the Lemmas 1-4 will be used to derive sufficient stability conditions to the uncertain system (2) in the form of LMIs.

4. ROBUST STABILITY ANALYSIS OF PWA SYSTEMS

This section presents the main result of the paper, which are sufficient conditions to assess the global exponential stability of the origin of the uncertain PWA system (2). Moreover, we formulate a numerical test, in the form of LMIs, to verify such conditions.

Theorem 5. (Robust stability analysis). If there exist matrices $P > 0 \in \mathcal{S}^n$, $M_i \in \mathcal{S}^{n_m}$, $\Lambda_{i,j} \in \mathcal{S}^{n_m}$, $T_i \in \mathcal{D}^m$, $R \in \mathbb{R}^{n_\xi \times m}$ and $\epsilon > 0 \in \mathbb{R}$ such that

$$-\bar{H}_i + \Pi_0^\top(\Psi(T_i) - M_i)\Pi_0 + \mathrm{He}(RQ_i) \geq 0 \tag{13a}$$
$$M_i - \Lambda_{i,j} \geq 0, \tag{13b}$$
$$\Lambda_{i,j(j,j)} = 0, \tag{13c}$$
$$\Lambda_{i,j(k,k)} + \Lambda_{i,k(k,j)} + \Lambda_{i,k(j,k)} = 0, \quad j \neq k, \tag{13d}$$
$$\Lambda_{i,j(k,\ell)} + \Lambda_{i,k(\ell,j)} + \Lambda_{i,\ell(j,k)} \geq 0, \quad j \neq k \neq \ell, \tag{13e}$$

are satisfied for $i = 1, \ldots, n_\theta$ and $j = 1, \ldots, n_m$, where

$$Q_i = [e_i \; C_i \; (D_i - I_m) \; I_m] \;\; \text{and}$$

$$\bar{H}_i = \mathrm{diag}\left(0, \begin{bmatrix} A_i^\top P + PA_i + \epsilon I_n & PB_i \\ B_i^\top P & 0_m \end{bmatrix}, 0_m\right),$$

with the vertices A_i, B_i, C_i, D_i and e_i defined by (3), then the origin of the uncertain system (2) is globally exponentially stable.

Proof: Since $P > 0$, $V(x)$ in (7) is positive-definite. Moreover, if the LMIs (13a) are satisfied, then it follows from Assumption 1 that

$$\sum_{i=1}^{n_\theta} \theta_i(-\bar{H}_i + \Pi_0^\top(\Psi(T_i) - M_i)\Pi_0 + \mathrm{He}(RQ_i)) \geq 0$$

for all $\theta \in \Theta$. Note that according to Lemma 4 the matrices M_i are copositive thanks to (13b)-(13e) and, therefore, any convex combination of M_i is copositive as well. Thus, by pre and post multiplying the above LMI for $\xi^\top(y(x))$ and $\xi(y(x))$, respectively, and applying Lemmas 1, 2 and 3 with

$$M(\theta) = \sum_{i=1}^{n_\theta} \theta_{(i)} M_i,\ T(\theta) = \sum_{i=1}^{n_\theta} \theta_{(i)} T_i \text{ and } Q(\theta) = \sum_{i=1}^{n_\theta} \theta_{(i)} Q_i,$$

we obtain that $\dot{V}(x) \leq -\epsilon x^\top x$ for any (possibly time-variant) $\theta \in \Theta$, from where the robust exponential stability of the origin of (2) follows. □

In the next section we illustrate the effectiveness of the proposed method with two numerical examples.

5. NUMERICAL EXAMPLES

This section provides two examples to illustrate the effectiveness of Theorem 5. The first one is a numerical example built to show some of the possible partition modifications induced by the uncertain parameters. The second one regards the stability analysis of a receding-horizon MPC controller.

5.1 Example 1

Consider an uncertain PWA system given by (2) with

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix},\quad B = \begin{bmatrix} -6 & 6 & 6 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C(\nu) = \begin{bmatrix} -0.15 & 0.15 \\ 0.15 & -0.15 \\ \nu_1 & \nu_2 \end{bmatrix},\quad D = 0_3 \text{ and } e = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix},$$

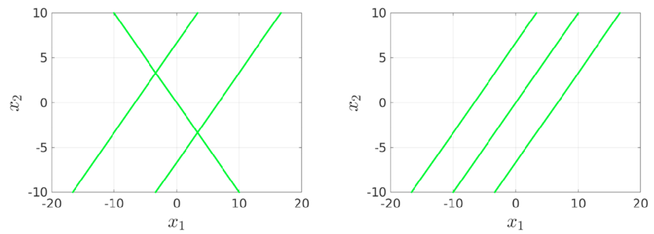
where $\nu_1 \in \kappa[-0.10,\ 0.05]$ and $\nu_2 \in \kappa[-0.02,\ 0.04]$, with $\kappa > 0$. The uncertain system can be written in terms of polytopic uncertainties, with the vertices of (3) given by $A_i = A$, $B_i = B$, $D_i = D$ and $e_i = e$, for $i = 1, \ldots, 4$, and

$$C_1 = \begin{bmatrix} -0.15 & 0.15 \\ 0.15 & -0.15 \\ -\kappa 0.10 & -\kappa 0.02 \end{bmatrix},\quad C_2 = \begin{bmatrix} -0.15 & 0.15 \\ 0.15 & -0.15 \\ \kappa 0.05 & -\kappa 0.02 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} -0.15 & 0.15 \\ 0.15 & -0.15 \\ -\kappa 0.10 & \kappa 0.04 \end{bmatrix} \text{ and } C_4 = \begin{bmatrix} -0.15 & 0.15 \\ 0.15 & -0.15 \\ \kappa 0.05 & \kappa 0.04 \end{bmatrix}.$$

To illustrate the influence of the uncertainties over the partition, Figure 1 shows the regions of the state-space for two sets of values of ν_1 and ν_2. Note that the system has six regions for $\nu_1 = -0.03$ and $\nu_2 = -0.03$ (Figure 1a) and four regions for $\nu_1 = -0.03$ and $\nu_2 = 0.03$ (Figure 1b).

To assess the global exponential stability of the origin of the system (2), the conditions of Theorem 5 were applied. The LMIs were solved using MATLAB, YALMIP (Lofberg, 2004) and MOSEK (2021), along with a line



(a) $\nu_1 = -0.03$ and $\nu_2 = -0.03$. (b) $\nu_1 = -0.03$ and $\nu_2 = +0.03$.

Fig. 1. Partition of the state-space for different values of ν_1 and ν_2.

search to determine the maximum feasible value of κ. The stability of the origin was certified for $\kappa = 1.6287$ by a Lyapunov function (7) with

$$P = \begin{bmatrix} 0.5437 & -0.2877 \\ -0.2877 & 0.3508 \end{bmatrix}.$$

Note that with this value of κ, the different partitions in Figure 1 can occur.

For comparison, we studied the robust stability of this example with $M_i \in \mathcal{S}_+^{n_m}$, that is, elementwise nonnegative matrices. This is equivalent to consider only the LMIs (13a) with the additional constraints $M_i \succeq 0$ for $i = 1, \ldots, n_\theta$. In this case, a maximum of $\kappa = 1.6270$ was obtained, illustrating that the proposed conditions are less conservative than the conditions previously presented in (Groff et al., 2019) and (Cabral et al., 2021). Indeed, as observed by Parrilo (2000), the constraints (13b)-(13e) are less conservative than searching for elementwise nonnegative matrices M_i.

5.2 Example 2

This example regards the stability analysis of a continuous-time linear system connected with a receding-horizon Model Predictive Control (MPC) state feedback through a Zero-Order Holder (ZOH) using the proposed framework.

Consider a continuous-time system given by

$$\dot{x} = A_c x + B_c u = \begin{bmatrix} -0.1 & 1.0 \\ -1.0 & -0.1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u, \tag{14}$$

where $u \in \mathbb{R}^{n_u}$ is the control input. To design an MPC state feedback control law, we first obtain a discrete-time system, considering a sampling period $T = 0.1$, leading to

$$x^+ = A_d x + B_d u = \begin{bmatrix} 0.9851 & 0.0988 \\ -0.0988 & 0.9851 \end{bmatrix} x + \begin{bmatrix} 0.1018 \\ 0.0447 \end{bmatrix} u.$$

We consider the following optimization problem that leads to the MPC over a prediction horizon of $N_h \geq 1 \in \mathbb{N}$ steps:

$$\begin{aligned} \min_U \ & \sum_{k=0}^{N_h-1} x_k^\top Q_x x_k + u_k^\top Q_u u_k, \\ \text{subject to} \quad & x_{k+1} = A_d x_k + B_d u_k, \\ & GU \leq E x_0 + w, \end{aligned} \tag{15}$$

where $Q_x \geq 0 \in \mathcal{S}^n$ and $Q_u > 0 \in \mathcal{S}^{n_u}$ are weighting matrices, $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^{n_u}$ are the state and input at the discrete-time instants, the vector $U^\top = \begin{bmatrix} u_0^\top & \ldots & u_{N_h-1}^\top \end{bmatrix} \in \mathbb{R}^{N_h n_u}$ is the vector representing the sequence of control inputs and matrices $G \in \mathbb{R}^{n_c \times N_h n_u}$, $E \in \mathbb{R}^{n_c \times n}$ and vector $w \in \mathbb{R}^{n_c}$ represent the n_c constraints of the optimization problem, which can be on the inputs, states or linear combinations of those. Once the optimal control input sequence U^* is found, its first input, that is, u_0^*, is applied to the system and a receding horizon policy is considered.

For this example, we considered $N_h = 3$, $Q_x = 5 \times I_2$, $Q_u = 1$, $G^\top = [I_3 \ -I_3]$, $E = 0_{6,2}$ and $w^\top = \left[3 \times \mathbf{1}_3^\top \ \mathbf{1}_3^\top\right]$. The matrices G, E and the vector w define constraints only on the input, with asymmetric bounds of $-1 \le u \le 3$. The feedback control law $u(x)$ that arises from (15) is a PWA function (Bemporad et al., 2002) and can be written in terms of the vector-valued ramp function ϕ as (Valmorbida and Hovd, 2022)

$$\begin{aligned} u(x) &= A_u x + B_u \phi(y(x)), \\ y(x) &= C_u x + D_u \phi(y(x)) + e_u, \end{aligned} \tag{16}$$

with

$$A_u = [-0.8256 \ -0.4626],$$

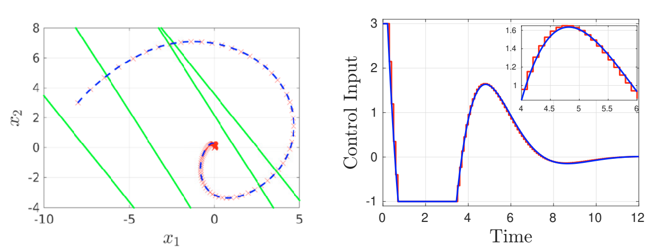
$$B_u = [-0.4469 \ 0.0256 \ 0 \ 0.4469 \ -0.0256 \ 0],$$

$$C_u = \begin{bmatrix} -0.8256 & -0.4626 \\ -0.3722 & -0.2691 \\ 0 & 0 \\ 0.8256 & 0.4626 \\ 0.3722 & 0.2691 \\ 0 & 0 \end{bmatrix},$$

$$D_u = \begin{bmatrix} 0.5531 & 0.0256 & 0 & 0.4469 & -0.0256 & 0 \\ 0.0256 & 0.5276 & 0 & -0.0256 & 0.4724 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.4469 & -0.0256 & 0 & 0.5531 & 0.0256 & 0 \\ -0.0256 & 0.4724 & 0 & 0.0256 & 0.5276 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \end{bmatrix}$$

and $e_u^\top = \left[-3 \times \mathbf{1}_3^\top \ -\mathbf{1}_3^\top\right]$.

We can use the above expression for $u(x)$ to write a continuous-time closed-loop PWA representation of the system under the MPC receding-horizon control policy. However, this representation is not exact, since the continuous-time evaluation of $u(x)$ does not match the discrete-time computed control law $u_{zoh}(x)$, as illustrated for a trajectory in Figure 2.



(a) Example of trajectory. (b) Control input.

Fig. 2. (2a) Example of trajectory of the closed-loop system for $x_0^\top = [-8 \ 3]$. Green lines represent the state-space partition. The blue line and the red crosses are, respectively, the continuous-time and the discrete-time trajectories. (2b) The blue line and the red line represent, respectively, the continuous-time evaluation of $u(x)$ and the discrete control law $u_{zoh}(x)$.

To analyse the stability of (14) controlled by $u_{zoh}(x)$ using the proposed framework, we compute $u_i(x)$ control laws, for $i, \ldots, n_\theta$, such that $u_{zoh}(x)$ can be written as a polytopic combination along the sampling period. The

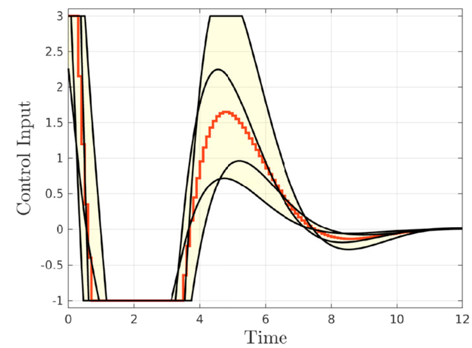


Fig. 3. Vertex control inputs (black lines) and its envelope (yellow area) encompassing the discrete control law $u_{zoh}(x)$ (red line) for the trajectory depicted in Fig. 2.

control laws $u_i(x)$ are obtained by modifying the matrices Q_x and Q_u in (15). Then, we evaluate $u_i(x)$ along the sampling period in a grid given by $|x_1| \le 10$ and $|x_2| \le 10$, with a grid step of 0.1, to verify that $u_{zoh}(x)$ can be written as a polytopic combination of $u_i(x)$ along the sampling period. To satisfy the inclusion of $u_{zoh}(x)$ in the polytope, we used $n_\theta = 4$. The representation for each of those control laws is given in the Appendix, while Figure 3 shows those control laws for a trajectory.

The robust exponential stability of the origin was certified by a Lyapunov function (7) with

$$P = \begin{bmatrix} 5.9338 & -0.0155 \\ -0.0155 & 6.3468 \end{bmatrix}.$$

6. CONCLUSION

This work addressed the problem of assessing the stability of the origin of uncertain PWA systems. To solve this problem, we extended the use of an implicit representation based on vector-valued ramp functions to *continuous-time* PWA systems. The main advantage of the proposed method is that the robust stability analysis can be performed for the case where the uncertainties modify both the shape of the partition and the number of regions. Numerical examples illustrate the effectiveness of the proposed method. Future work will tackle the robust stability analysis using Piecewise Quadratic (PWQ) Lyapunov candidate functions. The main challenge in the analysis with PWQ functions is its non-differentiability in the boundaries of the regions of the partition.

APPENDIX

This appendix contains details about the Example 2. The zero-order holder control law $u_{zoh}(x)$ can be written as a polytopic combination of four PWA continuous-time control laws (16) with

$$A_{u1} = \left[-0.9416 \ -0.1199\right], \quad A_{u2} = \left[-0.7496 \ -0.6871\right],$$

$$A_{u3} = \left[-1.7313 \ -0.9609\right], \quad A_{u4} = \left[-0.3582 \ -0.2015\right],$$

$$B_{u1} = \left[-0.5633 \ 0.0298 \ 0 \ 0.5633 \ -0.0298 \ 0\right],$$

$$B_{u2} = \left[-0.3703 \ 0.0223 \ 0 \ 0.3703 \ -0.0223 \ 0\right],$$

$$B_{u3} = \left[-0.9720 \ 0.1282 \ 0 \ 0.9720 \ -0.1282 \ 0\right],$$

$$B_{u4} = \left[-0.1908 \ 0.0045 \ 0 \ 0.1908 \ -0.0045 \ 0\right],$$

$$C_{u1} = \begin{bmatrix} -0.9416 & -0.1199 \\ -0.4235 & -0.1132 \\ 0 & 0 \\ 0.9416 & 0.1199 \\ 0.4235 & 0.1132 \\ 0 & 0 \end{bmatrix}, \quad C_{u2} = \begin{bmatrix} -0.7496 & -0.6871 \\ -0.3390 & -0.3702 \\ 0 & 0 \\ 0.7496 & 0.6871 \\ 0.3390 & 0.3702 \\ 0 & 0 \end{bmatrix},$$

$$C_{u3} = \begin{bmatrix} -1.7313 & -0.9609 \\ -0.7364 & -0.5529 \\ 0 & 0 \\ 1.7313 & 0.9609 \\ 0.7364 & 0.5529 \\ 0 & 0 \end{bmatrix}, \quad C_{u4} = \begin{bmatrix} -0.3582 & -0.2015 \\ -0.1654 & -0.1177 \\ 0 & 0 \\ 0.3582 & 0.2015 \\ 0.1654 & 0.1177 \\ 0 & 0 \end{bmatrix},$$

$$D_{u1} = \begin{bmatrix} 0.4367 & 0.0298 & 0 & 0.5633 & -0.0298 & 0 \\ 0.0298 & 0.4057 & 0 & -0.0298 & 0.5943 & 0 \\ 0 & 0 & 0.375 & 0 & 0 & 0.625 \\ 0.5633 & -0.0298 & 0 & 0.4367 & 0.0298 & 0 \\ -0.0298 & 0.5943 & 0 & 0.0298 & 0.4057 & 0 \\ 0 & 0 & 0.625 & 0 & 0 & 0.375 \end{bmatrix},$$

$$D_{u2} = \begin{bmatrix} 0.6297 & 0.0223 & 0 & 0.3703 & -0.0223 & 0 \\ 0.0223 & 0.6080 & 0 & -0.0223 & 0.3920 & 0 \\ 0 & 0 & 0.5833 & 0 & 0 & 0.4167 \\ 0.3703 & -0.0223 & 0 & 0.6297 & 0.0223 & 0 \\ -0.0223 & 0.3920 & 0 & 0.0223 & 0.6080 & 0 \\ 0 & 0 & 0.4167 & 0 & 0 & 0.5833 \end{bmatrix},$$

$$D_{u3} = \begin{bmatrix} 0.0280 & 0.1282 & 0 & 0.9720 & -0.1282 & 0 \\ 0.1282 & -0.0996 & 0 & -0.1282 & 1.0996 & 0 \\ 0 & 0 & -0.25 & 0 & 0 & 1.25 \\ 0.9720 & -0.1282 & 0 & 0.0280 & 0.1282 & 0 \\ -0.1282 & 1.0996 & 0 & 0.1282 & -0.0996 & 0 \\ 0 & 0 & 1.25 & 0 & 0 & -0.25 \end{bmatrix},$$

$$D_{u4} = \begin{bmatrix} 0.8092 & 0.0045 & 0 & 0.1908 & -0.0045 & 0 \\ 0.0045 & 0.8047 & 0 & -0.0045 & 0.1953 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0.2 \\ 0.1908 & -0.0045 & 0 & 0.8092 & 0.0045 & 0 \\ -0.0045 & 0.1953 & 0 & 0.0045 & 0.8047 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0.8 \end{bmatrix},$$

and $e_{u1}^\top = e_{u2}^\top = e_{u3}^\top = e_{u4}^\top = [-3\ -3\ -3\ -1\ -1\ -1]$. The closed-loop continuous-time vertex systems can be expressed, for $i = 1, \ldots, 4$, as

$$\dot{x} = (A_c + B_c A_{ui})x + B_c B_{ui}\phi(y_i(x)),$$
$$\phi(y_i(x)) = C_{ui}x + D_{ui}\phi(y_i(x)) + e_{ui}.$$

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